The Population Dynamics of File-Sharing Peer-to-Peer Networks

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***Abstract*: Recently peer-to-peer file-sharing applications have shown an extreme popularity and the workload generated to the Internet has been dominated by the traffic coming from these applications. In this paper we develop a simple but effective mathematical model to capture the file population dynamics of such systems. Our modeling framework is based on the theory of branching processes. We describe analytically the behavior of the proposed model. The precise characterization of the necessary and sufficient conditions of population extinction or explosion is given based on the system parameters. We also present the expected ratio of active, passive and dead peers for the long-term regime. We validate and demonstrate our results in several simulation studies. Based on our results we propose a number of engineering guidelines to the design and control of file-sharing P2P systems.**

Keywords: peer-to-peer networking, population dynamics, branching processes.

I. INTRODUCTION

Recent traffic measurements (e.g. [1]) show that the workload generated by P2P applications are the dominant part of most of the Internet segments. In spite of the fact that the popularity of current P2P applications changes fast, it seems that the file sharing-like applications were, are and probably will be the most popular application type among all the P2P applications. P2P file sharing also shows an evolution starting from Napster and going through many new developments resulted in Gnutella, Kazaa, Morpheus, eDonkey, BitTorrent, etc.

In this paper we analyze the population dynamics of a file-sharing peer-to-peer system. We build up a general model which is capable of capturing all the important characteristics of relevant P2P file-sharing systems. We perform a comprehensive performance analysis based on the theory of branching processes. We investigate the characteristics of the system and present several results about the necessary and/or sufficient conditions of extinction, stagnation and explosion of the population size of shared files. Our analytical results are validated by a simulation study and we also present a number of examples about the evolution of population size in different cases. Finally, we derive a number of useful engineering guidelines from the results which may help the design and the control of peer-to-peer file sharing systems.

The rest of the paper is organized as follows. We overview the relevant related work in Section I-A. Our general model with its parameter description is introduced in Section II. Our detailed analytical study with the main results is presented in Section III. The validation results with engineering implications are described in Section IV. Finally, we conclude our paper in Section V.

Some of the results in this paper have been published in a short form of a conference paper [14]. In this extended version we present details all our findings in this research approach, including further results in Section III/B, C and Section IV.

A. Related work

Most of the early P2P research was mainly focused on traffic measurements and design. These fields are still active and recently several studies were published reporting results on these areas with related characterization studies, e.g. [9]. On the other hand, the performance evaluation of P2P systems is becoming a hot topic of recent research. Starting from [10] where a closed queueing system is used to analyze the performance of a P2P system a number of new results were published trying to get some more understanding about the behavior of such systems. Focusing only on the topic of this paper papers [2-8] are the most closely related published results.

In [5] the authors studied the service capacity of a P2P system both in the transient regime with a branching process model and also in the stationary regime with a Markov chain model. They have found, among others, an exponential growth of service capacity during the transient phase. Several papers focused on the currently popular BitTorrent P2P applications, e.g. [2] [8] [4] [7]. The authors of [2] have applied a fluid model to reveal the performance and scalability aspects of BitTorrent. [8] presents an extensive trace analysis and modeling study of BitTorrent-like systems. The paper [4] uses a deterministic fluid model and a Markov chain to study the system behavior and an approximation for the life time of a chunk in BitTorrent is also proposed. The behavior of the peers in BitTorrent is studied in the paper [7], where the authors also investigate the file availability and the dying-out process. The population dynamics of the P2P systems is also addressed in [3], where a spatio-temporal model is proposed to analyze the resource usage of the system.

In the recent paper [15] authors propose a general population dynamics model for DS over P2P with fixed population. The dynamic of the peers is captured by a closed Markov queuing network and they prove that this model has equilibrium and only one closed-form solution. Wang *et al.* studied the evolutionary dynamics of reciprocity-based incentive mechanism, in P2P systems based on Evolutionary Game Theory (EGT). They found that the intensity of selection plays an important role in the evolutionary dynamics of P2P incentive mechanism [16]. Authors in [17] present an extensive study of BitTorrent availability through measurement and analysis. They show, among other things, that the variability of availability shows a typical life cycle pattern over time implying that it is difficult for users to obtain files in the latter half of stage.

The main difference between these papers and our paper is that our analysis is entirely based on the theory of branching processes. We create a reasonable model for filesharing P2P system and derive a detailed characterization of the system in a particular way. In the most related previous work the authors in [5] also applied a branching process model but their analysis was restricted to showing the sensitivity of the exponential growth behavior to the system parameters in the transient regime.

II. MODELING FILE POPULATION OF P2P SYSTEMS

The objective of the proposed model is to describe the main characteristics of P2P file-sharing systems: the population of shared files. Technically, all the available P2P file-sharing systems apply the same rule. P2P users contribute to the common system resource by providing the access to a set of their files and they have access to the common resource in return. In general, the common resource consists of one or several copies of some individual, unique files. This is straightforward since a unique file provided by a P2P user in the file-sharing system will be downloaded by the other peers and some of them will also share this one to the system. The system state is modified each time a file download is completed.

From a modeling point of view the operation of P2P systems can be simplified by focusing on an individual file. At a certain point in time, a file is first introduced into the system. Assuming that this individual is ’interesting’ for the community: it could be a new movie video or a popular MP3 song. Probably this file will be downloaded by some other peers and now there are already several copies of the file in the system. The new copy can also be cloned by further peer’s download and so on. This mechanism is very similar to the branching process model of population growth, mainly applied in the field of biology. This suggests the idea of using branching processes to model the file population of the P2P file-sharing systems. Results from the analysis of branching processes can give us a detailed understanding of the population size of shared files, which is the most important feature of a P2P system. The conditions related to the explosion, stagnation or extinction of the population could be the milestones of a successful P2P system design.

Branching processes have been studied for over a century. The applications of branching processes are found in many areas such as population dynamics, algorithms, molecular biology, etc. The simplest single type discrete time branching model is presented in the next section.

A. Branching Processes

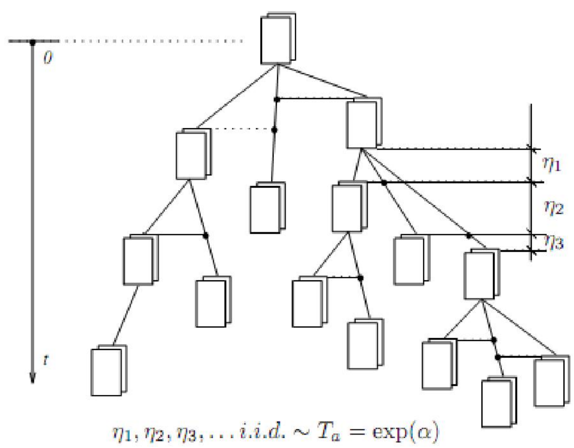


Figure 1. Branching process-like file replications in the P2P system

Suppose that at the beginning there are X_0 individuals. In every generation each individual independently gives rise to a number of offsprings. Denote by $\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{X_n}^{(n)}$ the number of offsprings of X_n individuals in the nth generation. $\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{X_n}^{(n)}$ are i.i.d. random variables having the distribution:

$$P[\xi = k] = pk,\ k = 0, 1, 2, \ldots \qquad (1)$$

The total size of the population in the $(n+1)$st generation is

$$X_{n+1} = \xi_1^{(n)} + \xi_2^{(n)} + \ldots + \xi_{X_n}^{(n)}. \qquad (2)$$

The sequence $\{X_n\}_0^\infty$ is called a branching process with initial population size X_0 and offspring distribution $\{p_j\}$. The definition of branching process assumed that X_n is independent of $\xi_k^{(m)}$ for all m,k.

Denote the mean and variance of the number of offsprings of an individual by $\mu=[\xi]$ and $\sigma^2=[\xi]$. It can be shown, e.g. in [1], that the mean and variance of the population size in the nth generation, denoted by $M(n)=[X_n]$ and $V(n)=[X_n]$, satisfy

$$M(n) = \mu M(n-1)$$
$$V(n) = \sigma^2 M(n-1) + \mu^2 V(n-1).$$

By iterating, we have

$$M(n) = \mu^n M(0). \qquad (3)$$

If we impose that $X_0=1$ then

$$M(n) = \mu^n$$
$$V(n) = \sigma^2(\mu^{n-1} + \mu^n + \ldots + \mu^{2n-2}).$$

The mean of offsprings (μ) has direct impact on the behavior of population growth: extinction or explosion. The branching processes with $\mu<1$, $\mu=1$, and $\mu>1$ are referred to as *subcritical*, *critical*, and *supercritical* branching processes, respectively. In the first two cases the popularity dies out with probability 1, while in the last case the population size tends to ∞ as n increases. An important difference between the subcritical and critical cases is that the mean of the extinction time $T=\min\{n\geq 1 : X_n=0\}$ is finite for $\mu<1$, i.e., $E[T]<\infty$, and infinite for $\mu=1$. Note that in both cases $P[T<\infty]=1$ [1].

B. Age-Dependent Multitype Branching Process Model of File Population

The real operation of the P2P system is much more complicated than the model discussed above. Several important properties of real P2P systems must be taken into account:

- Offsprings (copies) of an individual (file) are born at different (random) points in time.
- Free riding problem: there is always a group of peers who download files without contributing to the system by making their files accessible to the others. The offsprings owned by these peers will have no descendants. They are considered 'dead' from the point of view of the system.
- Peers possessing the concerned file may not share the file constantly. Sometimes they can be offline when downloads from that peer are not possible.

In addition, a file can be downloaded in some parts from several peers having the same file in the P2P system. It means that several similar individuals may contribute to the origin of an offspring. Furthermore, peers can even share incomplete objects in some P2P applications. However, this kinds of births have little impact on the overall population of the system in general. Therefore we assume only single and complete parent model in our description. In other words, a file can be shared only if it is complete and it has the origin from only one peer.

Combining these characteristics we propose a model of age-dependent multitype branching process for P2P file-sharing system. We differentiate between two types of peers owning the concerned file: cooperating peers and free riders. After the successful download of a file the cooperating peers will share the file with the system, contributing to the newer copies of the file in the system. Further, cooperating peers have two possible states, active (A) and passive (P), corresponding to their online and offline activities. An online peer can give rise to a new copy while offline peers are unaccessible, thus do not create new offsprings. Non-cooperating peers or free riders are considered as dead (D) peers (copies), since they do not contribute to the birth of any offsprings. The possible transitions between states and the corresponding probabilities are shown in Figure 2.

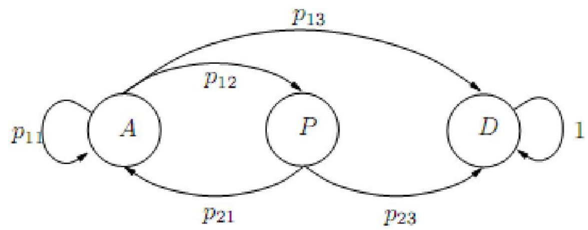


Figure 2. The state transitions and probabilities

We assume that an active peer can only change its state when the offspring is born. To be more specific, when an individual is born, it has to choose to be active, passive or non-cooperative, i.e. "dead". If the individual is active, it will stay in this state until its offspring is born. State transitions of active peers only happen at these instants.

In the model we use the following assumptions and notations:

- The age time, i.e. the age of the parent when offspring is born, is a random variable T_a with an exponential distribution with mean α. A peer can have several offsprings during their activity time (lifetime) in the system. The age of the parent is counted from the parent's activation time, i.e. when it turns from passive to active or it is just born, or from the birth of the last born offspring (see Figure 1).
- The offline time of a peer (length of passive period) is also an exponential variable T_p with mean β.
- The expected value of offsprings in a single birth is λ. This parameter expresses the average number of *exact* parallel downloads from a peer. In the P2P case λ≡1, since the probability that multiple downloads of a file from a peer end at the same time is zero. However, we use λ in the general discussion of the model.
- Let $\{\pi_i\}_{i=1}^{3}$ be the probabilities that a new peer becomes active, passive or dead. Clearly, $\sum_{i=1}^{3}\pi_i=1$.

From this point the lower indices 1,2,3 will refer to active, passive and dead peer states.

- The type-transition matrix which describes the probabilities of state transitions is the following:

$$\begin{array}{c} \\ A \\ P \\ D \end{array} \begin{array}{c} \begin{array}{ccc} A & P & D \end{array} \\ \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & 0 & p_{23} \\ 0 & 0 & 1 \end{bmatrix} \end{array},$$

where A,P and D stand for active, passive and dead state, respectively. For example: P[$active \rightarrow$ dead]=p_{13}

Obviously, $\sum_{j=1}^{3} p_{1j}=1$, $p_{21}+p_{23}=1$ since the type-transition matrix is a stochastic matrix.

With the assumptions of the memoryless property of the age time T_a and the offline time T_p the population size process is Markovian. The next section derives the expected size of the population and the most important features of the process.

III. ANALYSIS OF THE MODEL

A. Model description

In this subsection we show how our branching process model can be characterized by its transition operator and we derive the operator parameters from our P2P system model parameters.

Theoretically, if the generating function [1] of a branching process is known, then all important properties of the process is determined (e.g. the extinction probability, the expected population size, the deviation of size). Unfortunately, to determine the generating function of the proposed branching model, the following probabilities should be calculated:

P[the number of active and passive peers at time t_0 is (k, l)], which is a very complicated task. Therefore we avoid the use of generating functions.

Let $Z_t=(Z_t^{(1)},Z_t^{(2)},Z_t^{(3)})$ be the vector representing the population size of active, passive and dead peer at the time instant t. Let M_t be the transition operator defined by:

$$Z_t=M_tZ_0.$$

It is easy to see that the process we investigate depends linearly on Z_0, which means that M_t is a random matrix.

First of all, since the process is Markovian it can be realized that the following equation holds:

$$M_{t+s}=M_tM_s, \qquad (4)$$

where M_t and M_s are independent random matrices. This implies:

$$\lim_{n\to\infty}(EM_{t/n})^n = EM_t \qquad (5)$$

The element $(EM_t)_{1,1}$ of the matrix EM_t can be determined as follows. Let us choose an appropriate small time interval $\delta\in R^+$, such that the probability that two or more downloads are finished in δ is $o(\delta)$, where $\lim_{\delta\to 0}\frac{o(\delta)}{\delta}=0$. Thus the probability that an active peer is going to have children within the time interval δ is $\frac{\delta}{\alpha}+o(\delta)$. Similarly, we have:

$$\text{P[a passive peer becomes active within } \delta] = \frac{\delta}{\beta}+o(\delta)$$

The average number of active peers produced by one active peer after the time interval δ is $\lambda\pi_1\frac{\delta}{\alpha}$ because there are λ offsprings on average and only $\lambda\pi_1$ will be active. But an active peer may also become passive or dead with probabilitie p_{12} and p_{13}; the probability that no file-sharing happens is $1-\frac{\delta}{\alpha}+o(\delta)$. Then,

$$(\mathbf{E}M_\delta)_{1,1} = \frac{\delta}{\alpha}\lambda\pi_1 + (1-p_{12}-p_{13})\frac{\delta}{\alpha} + \left(1-\frac{\delta}{\alpha}\right) \qquad (6)$$

Using similar considerations M_δ is given by:

$$\mathbf{E}M_\delta = \begin{bmatrix} \frac{\delta}{\alpha}\lambda\pi_1 + \frac{\delta}{\alpha}p_{11} + (1-\frac{\delta}{\alpha}) & \frac{\delta}{\beta}p_{21} & 0 \\ \frac{\delta}{\alpha}\lambda\pi_2 + \frac{\delta}{\alpha}p_{12} & (1-\frac{\delta}{\beta}) & 0 \\ \frac{\delta}{\alpha}\lambda\pi_3 + \frac{\delta}{\alpha}p_{13} & \frac{\delta}{\beta}p_{23} & 1 \end{bmatrix} + o(\delta)$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} + \delta \underbrace{\begin{bmatrix} \frac{\lambda\pi_1+p_{11}-1}{\alpha} & \frac{p_{21}}{\beta} & 0 \\ \frac{\lambda\pi_2+p_{12}}{\alpha} & -\frac{1}{\beta} & 0 \\ \frac{\lambda\pi_3+p_{13}}{\alpha} & \frac{p_{23}}{\beta} & 0 \end{bmatrix}}_{A} + o(\delta)$$

Letting $\delta=\frac{t}{n}$, we get

$$\mathbf{E}M_t = \lim_{n\to\infty} \left(\mathbf{E}M_{t/n}\right)^n =$$

$$= \lim_{n\to\infty} \left(I + \frac{At}{n}\right)^n = \exp(At) \qquad (7)$$

Eq. (7) implies that EZ_t grows exponentially with a rate determined by the eigenvalues of A. Let γ_i, i = 1, 2, 3 be the eigenvalues of A. It is clear that one of them is zero. Put $\gamma_3=0$. The other two eigenvalues of A are given by:

$$\lambda_{1,2} = \frac{a_{11}+a_{22}\pm\sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}}{2}$$

$$:= \frac{-b\pm\sqrt{b^2-4c}}{2}, \qquad (8)$$

where a_{ij} is the (i,j)-th element of A,

$$b:=-(a_{22}+a_{11})=\frac{1}{\beta}-\frac{\lambda\pi_1+p_{11}-1}{\alpha},$$

and

$$c:=a_{11}a_{22}-a_{12}a_{21}=\frac{(1-\lambda\pi_1-p_{11})-p_{21}(\lambda\pi_2+p_{12})}{\alpha\beta}.$$

Considering Eq. (8) one can clearly see that $\lambda_{1,2}$ are real numbers since $a_{1,2}, a_{2,1} \geq 0$. In addition, $\gamma_1=\gamma_2$ if and only if

$$\underbrace{(a_{11}-a_{22})^2}_{\geq 0} = \underbrace{-4a_{12}a_{21}}_{\leq 0}$$

$$\Leftrightarrow \begin{cases} a_{11} = a_{22} = -\frac{1}{\beta} < 0 \\ p_{21} = 0 \quad \text{or} \quad (\pi_2 = 0 \text{ and } p_{12} = 0) \end{cases}, \qquad (9)$$

i.e., $\gamma_1=\gamma_2<0$ and either $p_{12}=0$ or $p_{21}=0$. This means that either active peers cannot become passive or passive ones cannot become active. This is completely unlikely and unrealistic regarding the concerned systems. Thus the investigation of this case is ignored. Put $\gamma_1>\gamma_2$.

B. *The expected population size of the process*

Using the model description presented above some important properties of the system can be derived. In this subsection we present several necessary and/or sufficient conditions of extinction, stagnation and explosion of the population size of shared files. The ratio of active, passive and dead peers in the long-term behavior is also provided.

Since the maximal eigenvalue of A determines the behavior of the process, it is worth differentiating between two cases:

- $\max\{\gamma_i\}>0$, i.e. there exists at least a positive eigenvalue of A (so $\gamma_1>0$).
- $\max\{\gamma_i\} = 0$, i.e. $\gamma_{1,2} \leq 0$

By Eq. (8) it is straightforward that the first case happens if $b<0$ or ($b\geq 0$ *and* $c<0$). These two conditions can be expressed regarding the expected number of offsprings of a single birth λ as

$$b < 0 \quad \Leftrightarrow \quad \lambda > \frac{\alpha + \beta(1-p_{11})}{\beta\pi_1} \qquad (10)$$

$$(b \geq 0 \;\&\; c < 0) \;\Leftrightarrow \begin{cases} \frac{\alpha+\beta(1-p_{11})}{\beta\pi_1} \geq \lambda \\ \lambda > \frac{(1-p_{11})+p_{12}p_{21}}{\pi_1+\pi_2 p_{21}} \end{cases} \qquad (11)$$

The latter condition in Eq. (11) is fulfilled if

$$\frac{\alpha + \beta(1-p_{11})}{\beta\pi_1} > \frac{(1-p_{11})+p_{12}p_{21}}{\pi_1+\pi_2 p_{21}} \qquad (12)$$

$$\Leftrightarrow \quad \frac{\alpha}{\beta} \quad > \quad p_{21}\frac{p_{12}(\pi_1-\pi_2)-\pi_2 p_{13}}{\pi_1+\pi_2 p_{21}} \quad (13)$$

The left-hand side of Eq. (13) is always positive, so the inequality holds if the right hand side is non-positive, i.e.,

$$p_{12}(\pi_1-\pi_2)-\pi_2 p_{13} \le 0 \quad \Leftrightarrow \quad \frac{\pi_1}{\pi_2} \le 1+\frac{p_{13}}{p_{12}}. \quad (14)$$

Summarizing the results we get

Lemma 1: If the condition $\left(\frac{\pi_1}{\pi_2} \le 1+\frac{p_{13}}{p_{12}}\right)$ holds, the matrix A has positive eigenvalue(s) if and only if

$$\exists i{:}\gamma_i>0 \quad \Leftrightarrow \quad \lambda > \frac{(p_{12}+p_{13})+p_{12}p_{21}}{\pi_1+\pi_2 p_{21}} \quad (15)$$

Note that $\lambda=1$ in our particular model.

The next statement provides the sufficient conditions for the existence of positive eigenvalues of *A*.

Lemma 2: There exists a positive eigenvalue of A if any of the following two conditions hold:

(i) $\lambda > \frac{(1-p_{11})+p_{12}p_{21}}{\pi_1+\pi_2 p_{21}}$

(ii) $\lambda=1$, $1-p_{11} \le \min\{\pi_1,\pi_2\}$, and $p_{13} \neq 0$

Proof:

(i) If Eq. (10) holds, then there must be a positive eigenvalue. Otherwise, the first part of Eq. (11) must hold. This one and the condition *(i)* together imply that both parts of (11) hold.

(ii) The conditions of *(i)* are simplified in our P2P model, when $\lambda=1$. Assume that $p_{21}, p_{12} \neq 0$, so $p_{11}<1$, and $\min\{\pi_1,\pi_2\}>0$. We will show that *(i)* holds.

$$\frac{(1-p_{11})+p_{12}p_{21}}{\pi_1+\pi_2 p_{21}} \le \frac{(1-p_{11})+p_{12}p_{21}}{\min\{\pi_1,\pi_2\}(1+p_{21})} =$$
$$= \frac{p_{12}}{\min\{\pi_1,\pi_2\}} + \frac{p_{13}}{\min\{\pi_1,\pi_2\}(1+p_{21})}$$
$$< \frac{p_{12}+p_{13}}{\min\{\pi_1,\pi_2\}} = \frac{1-p_{11}}{\min\{\pi_1,\pi_2\}} \le 1 = \lambda$$

Lemma 1 and 2 show the conditions for the existence of positive eigenvalue(s). These results are important since we will show later in Lemma 6 that the existence of positive eigenvalues results in the explosion of the population size. It is interesting that the sufficient conditions do not depend on several parameters, e.g. α, β, p_{12}, and also p_{21} in Lemma 2 *(ii)*.

The following lemma shows conditions for non-positive eigenvalues.

Lemma 3: If $\lambda=1$ and the following conditions hold then A has only non-positive eigenvalues, which implies that the population will stop growing with probability 1 (see Lemma 5):

$$\left.\begin{array}{l} p_{12} \ge \max\{\pi_1,\pi_2\} \\ 1-p_{11} \ge \pi_1 \end{array}\right\} \Rightarrow \gamma_i \le 0 \quad \forall i \quad (16)$$

Proof:

Using Eq. (10) and (11) one can observe that if

$$\lambda \le \min\left\{\frac{(1-p_{11})+p_{12}p_{21}}{\pi_1+\pi_2 p_{21}}, \frac{\alpha+\beta(1-p_{11})}{\beta\pi_1}\right\} \quad (17)$$

then *A* does not have any positive eigenvalues. It can be shown that the conditions in the lemma satisfy the inequality above since

$$\frac{(1-p_{11})+p_{12}p_{21}}{\pi_1+\pi_2 p_{21}} \ge \frac{(1-p_{11})+p_{12}p_{21}}{\max\{\pi_1,\pi_2\}(1+p_{21})} =$$
$$= \frac{p_{12}}{\max\{\pi_1,\pi_2\}} + \frac{p_{13}}{\max\{\pi_1,\pi_2\}(1+p_{21})}$$
$$\ge \frac{p_{12}}{\max\{\pi_1,\pi_2\}} \ge 1 = \lambda,$$

and $\frac{\alpha+\beta(1-p_{11})}{\beta\pi_1} \ge \frac{1-p_{11}}{\pi_1} \ge 1 = \lambda.$

If $\lambda_{1,2} \neq 0$ the eigenvectors of A associated to the eigenvalues $\gamma_{1,2,3}$ are the followings:

$$\mathbf{s_1}=\begin{pmatrix} \frac{p_{21}}{\beta} \\ \gamma_1 - \frac{\lambda\pi_1 + p_{11} - 1}{\alpha} \\ \frac{p_{23}}{\beta} + \frac{a_{31}a_{12} - a_{11}a_{32}}{\gamma_1} \end{pmatrix}; \tag{18}$$

$$\mathbf{s_2}=\begin{pmatrix} \frac{p_{21}}{\beta} \\ \gamma_2 - \frac{\lambda\pi_1 + p_{11} - 1}{\alpha} \\ \frac{p_{23}}{\beta} + \frac{a_{31}a_{12} - a_{11}a_{32}}{\gamma_2} \end{pmatrix}; \mathbf{s_3}=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since $\gamma_1 > \gamma_2$, the three eigenvectors form a basis in R^3 and the expected number of peers will be the following:

$$[Z_t]=\exp(At)Z_0=\sum_{i=1}^{3} c_i e^{\gamma_i t}\mathbf{s_i}, \tag{19}$$

Where c_i, $i=1,2,3$ are given as the solution of the equation system $Z_0 \equiv (Z_0^{(1)}, Z_0^{(2)}, Z_0^{(3)}) = \sum_{i=1}^{3} c_i \mathbf{s_i}$. Z_0 is the initial state of the system.

If $\gamma_1=0$ or $\gamma_2=0$ (only one of them can be zero) A has only two eigenvectors, the previous calculation is not valid in this case. However, it can be shown that the rank of A^k is 1 if $k \geq 2$, i.e.,

$$A^k=(\mathbf{x}\mathbf{y}^T)^k=\mathbf{x}(\mathbf{y}^T\mathbf{x})^{k-1}\mathbf{y}^T \tag{20}$$

where

$$\mathbf{x}=\begin{pmatrix} \frac{p_{21}}{\beta} \\ -\frac{1}{\beta} \\ \frac{a_{31}a_{12}+a_{32}a_{22}}{a_{11}+a_{22}} \end{pmatrix} \quad \mathbf{y}=\begin{pmatrix} \frac{\beta}{p_{21}} \frac{\lambda\pi_1 + p_{11} - 1}{\alpha} \\ 1 \\ 0 \end{pmatrix}$$

and $\mathbf{y}^T$ is the transpose of $\mathbf{y}$. Since the third eigenvalue of A is $\mathbf{y}^T\mathbf{x}=a_{11}+a_{22}$, clearly it is not zero.

$$\exp(At) = I+At+\sum_{k=2}^{\infty} \frac{(\mathbf{x}\mathbf{y}^T)^k t^k}{k!}$$

$$= I+At+\mathbf{x}\frac{1}{\mathbf{y}^T\mathbf{x}}\left[\sum_{k=2}^{\infty} \frac{(\mathbf{y}^T\mathbf{x}t)^k}{k!}\right]\exp(\mathbf{y}^T\mathbf{x}t)-1-\mathbf{y}^T\mathbf{x}t\,\mathbf{y}^T$$

$$= I+t(A-\mathbf{x}\mathbf{y}^T)_L+\mathbf{x}\mathbf{y}^T\frac{\exp(\mathbf{y}^T\mathbf{x}t)-1}{\mathbf{y}^T\mathbf{x}} \tag{21}$$

where

$$L = \frac{a_{11}a_{32} - a_{31}a_{12}}{a_{11} + a_{22}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{a_{22}}{a_{12}} & 1 & 0 \end{pmatrix} \tag{22}$$

It is easy to see that L exerts an influence only on the number of dead peers.

Summarizing the results:

Proposition 1: The expected value of file population at time t is given by

$$\mathbf{E}[Z_t] = \exp(At)Z_0 = \tag{23}$$
$$\begin{cases} \sum_{i=1}^{3} c_i e^{\gamma_i t}\mathbf{s_i} & \text{if } \gamma_1, \gamma_2 \neq 0 \\ Z_0 + t(LZ_0) + \mathbf{x}(\mathbf{y}^T Z_0)\frac{1}{\gamma_2}(e^{\gamma_2 t} - 1) & \text{if } \gamma_1 = 0 \\ Z_0 + t(LZ_0) + \mathbf{x}(\mathbf{y}^T Z_0)\frac{1}{\gamma_1}(e^{\gamma_1 t} - 1) & \text{if } \gamma_2 = 0 \end{cases}$$

Note that $\mathbf{x}=\mathbf{s_1}$ if $\gamma_2=0$ and $\mathbf{x}=\mathbf{s_2}$ if $\gamma_1=0$.

This yields some important results:

Lemma 4: If $\gamma_1=0$ the expected numbers of active and passive offsprings are bounded while the expected value of dead offsprings grows linearly as t tends to infinity.

Lemma 5: If $\gamma_1<0$, the process will stop growing with probability 1.

Proof:

Denote by $\|\mathbf{x}\|_1 := \sum_{i=1}^{3} x^{(i)}$ the sum of the components of the vector $\mathbf{x}$ (L^1 norm). Let

$\|Z_\infty\|_1 = \lim_{t\to\infty}\|Z_t\|_1$ (it exists because the total number of peers is monotonic increasing). Since $\gamma_1<0$, by Eq. (23) $\mathbf{E}\|Z_t\|_1$ is bounded.

If a monotonic increasing sequence of integers never stops growing then it has to tend to infinity. Further, applying the Markov inequality we get

$$\begin{aligned} & \mathbf{P}[\text{the process will never stop growing}] \\ = \;& \mathbf{P}[\|Z_\infty\|_1 = \infty] = \lim_{x\to\infty}\mathbf{P}[\|Z_\infty\|_1 > x] \\ \le \;& \lim_{x\to\infty}\frac{\mathbf{E}[\|Z_\infty\|_1]}{x} = \lim_{x\to\infty}\frac{\lim_{t\to\infty}\|\mathbf{E}[Z_t]\|_1}{x} = 0 \end{aligned}$$

Lemma 5 shows that in case of the existence of non-positive eigenvalues the population will become extinct. It is also interesting that the sufficient conditions in Lemma 3 do not depend on α, β, and p_{21}.

Lemma 6: If the matrix A has a positive eigenvalue, i.e. $\gamma_1>0$, the mean number of active, passive and also dead peers tends to infinity. However, the process can still die out in this case, even with a very small probability.

Proof:

See the proof of Prop. 2.

This lemma shows that the existence of the positive eigenvalue which determined by the parameters as shown by Lemma 1 and 2 yields to the explosion of population size of shared files.

Proposition 2: The proportion of active, passive and dead peers converges to a deterministic vector, namely

$$\lim_{t\to\infty}\frac{Z_t}{\|Z_t\|_1} = \quad (24)$$

$$\begin{cases} \frac{\mathbf{s}_1}{\|\mathbf{s}_1\|_1} & \text{if } \gamma_1>0 \text{ and } \lim_{t\to\infty}\|Z_t\|_1=\infty \\ (0,0,1) & \text{if } \gamma_1\le 0 \text{ or } \lim_{t\to\infty}\|Z_t\|_1<\infty \end{cases}$$

Proof:

It is already shown that if $\gamma_1<0$ then the process stops growing with probability 1. This implies that at the end there must be only dead peers since both active and passive peers may introduce new peers into the system and thereby raise the whole number of peers.

If $\gamma_1=0$ let B denote the event that $\frac{Z_t}{\|Z_t\|_1}$ does not tend to (0,0,1). Suppose P(B)>0. Let $\|Z_\infty\|_1 = \lim_{t\to\infty}\|Z_t\|_1$ (it exists since $\|Z_t\|_1$ is monotonically increasing). Our assumption implies $B\subseteq\{\|Z_\infty\|_1=\infty\}$ (if $Z_\infty<\infty$ then the process will become extinct almost surely).

$$\begin{aligned} (0,0,0) \ne & \lim_{t\to\infty}\int_{\{\|Z_\infty\|_1=\infty\}}\left[\frac{Z_t}{\|Z_t\|_1}-(0,0,1)\right]\frac{\|Z_t\|_1}{\mathbf{E}\|Z_t\|_1}d\mathbf{P} \\ = & \underbrace{\lim_{t\to\infty}\left[\frac{\mathbf{E}Z_t}{\mathbf{E}\|Z_{t_n}\|_1}-(0,0,1)\right]}_{(0,0,0)} - \\ & -\underbrace{\sum_{k=0}^{\infty}\lim_{t\to\infty}\int_{\{\|Z_\infty\|_1=k\}}\left[\frac{Z_t}{\|Z_t\|_1}-(0,0,1)\right]\frac{\|Z_t\|_1}{\mathbf{E}\|Z_t\|_1}d\mathbf{P}}_{(0,0,0)} \\ = & (0,0,0) \end{aligned}$$

First we applied our assumption and the fact that the point (0,0,1) in the convex set S={x∈R3| ||x||1=1 x≥0} is extremal (x∈S is extremal if $\overline{\exists} x_1,x_2,\ldots,x_N\in S$, $\overline{\exists}\lambda_1,\lambda_2,\ldots,\lambda_N$ such that $0\le\lambda_i\le 1$ ∀i and $x=\sum_{i=1}^{N}\lambda_i x_i$ where S is a convex set). This means that if $\frac{Z_t}{\|Z_t\|_1}$ does not tend to (0,0,1) then its mean also does not to (0,0,1). Our indirect assumption is proved to be wrong.

If $\gamma_1>0$ and the mean number of the population size of the process tends to infinity, the growing rate of Z_t is determined by $c_1e^{\gamma_1 t}\mathbf{s}_1$ because γ_1 is the largest eigenvalue of A (see Eq. (23)). It is important to note that the population size is likely to grow to infinity, but with probability less than 1. The reason is that there exists a very small but positive probability that all alive peers (active and passive) decide to die out at the same time. In addition, the proportion of the mean number of different peer types is expressed by the elements of $\mathbf{s}_1$. The probability that the limit holds is also less than 1 and it strongly depends on the number of initial peers. Since the elements of $\mathbf{s}_1$ are non zero, the number of all three types of peers separately tend to infinity.

C. The non-Markov case

Without the exponential assumptions of age time and offline time the branching model is not Markovian. However, our conjuncture is that the main features of the process do not change. The intuitive discussion of this case is given below.

We have proven in the Markovian case that the mean of the transition matrix $M_t=\exp(At)$ for some matrix A, which can be also rewritten as:

$$\mathrm{E}M_t = Ce^{\gamma t} + B(t) \qquad (25)$$

where C is the orthogonal projection to the subspace generated by the eigenvector belonging to the maximal eigenvalue of A (in our case this subspace was $\mathbf{s_1}$ in the case $\gamma_1>0$ and (0,0,1) if $\gamma_1<0$) and $B(t)$ is a remaining term (matrix). It is easy to see that $||B(t)|| = O(e^{\theta}t)$ for some $\theta < \gamma$. This result is similar to the following theorem from branching process theory for the general, non-Markov cases:

Theorem 1 [1] It can be shown that there exists a constant matrix C and a matrix B(t) such that

$$\mathrm{E}M_t = Ce^{\gamma t} + B(t) \qquad (26)$$

where $||B(t)|| = O(e^{\theta}t)$ for some $\theta < \gamma$.

The next result can also be expressed:

Theorem 2 [1] In the supercritical case, let $\gamma>0$ be the Malthusian parameter and $\mathbf{W}(t)=e^{-\gamma t}Z_t(=e^{-\gamma t}M_tZ_0)$. At certain further conditions one can prove that

$$\lim_{t\to\infty}\mathbf{W}(t)=\mathbf{v}W \qquad \text{a.s.}$$

$\mathbf{v}$ being the right eigenvector of the matrix A associated with the maximal eigenvalue, and W being a one-dimensional random variable.

Our model, on one hand, does not satisfy the conditions of the theorem since it is not a pure supercritical process. On the other hand, it has similar properties in the case $\gamma_1>0$: $\mathbf{s_1}$ corresponds to the vector $\mathbf{v}$. We do not investigate the meaning of W since it is just a constant factor which does not change the behavior of the modeled process. The only difference is that in our model the limit holds with probability less than 1. Thus it can be expected that this theorem 'partly' holds even if our process is not Markovian: there is a limit which determines the proportion of the peers and this limit exists with probability less than 1.

IV. RESULT VERIFICATION AND IMPLICATIONS

We have implemented a simulation study to verify the results presented in the previous sections. The simulation results are shown in this section. In addition, the implications of the results and some engineering guidelines are also described and discussed.

A. Simulation results

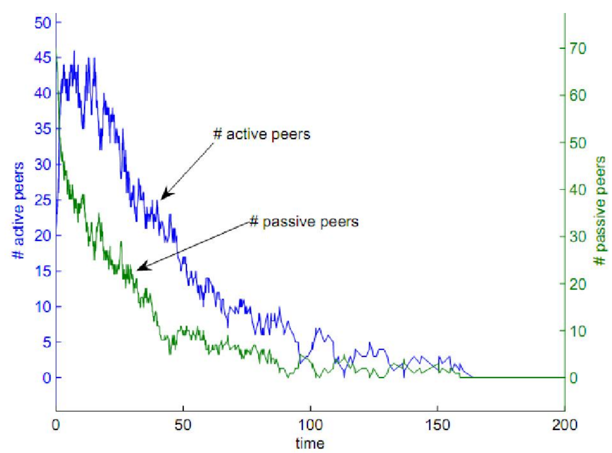


Figure 3. The case of population extinction

The P2P system model as described in section II-B is simulated using Matlab. The parameter λ is set to be 1, i.e., only an offspring is born from its parent at a certain point in time. We will show how the set of system parameters can predict exactly the long-term behavior of the P2P system.

Set α=4, β=5, $\pi_1 = \pi_2 = 0.05$; $p_{11} = 0.25$, $p_{12} = 0.5$, $p_{21} = 18/19$. It is easy to calculate that the conditions of Lemma 5 are satisfied. This means that the matrix A has only non-positive eigenvalues, i.e. the file population dies out almost surely. The exact values of γ_1 and γ_2 are calculated to be -0.025 and -0.349, respectively. The simulation result is shown in Figure 3. The figure displays the change of active and passive peers in the function of time. Once the number of these peers is zero the system is extinguished. It can be seen that it happens after about 160 time units.

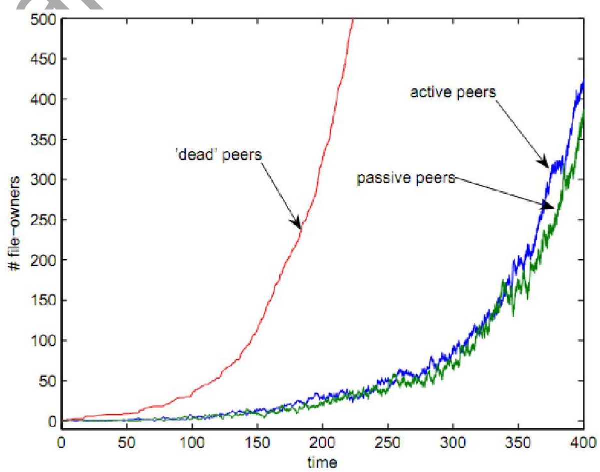


Figure 4. The case of population explosion

If we change π_1=0.15, π_2=0.25 while keeping the others unchanged the Lemma 2 holds. This implies that the population of all types of peers is likely to tend to infinity. The exact calculation provides γ_1=0.015,γ_2=−0.365. The growth of population is also justified by simulation results, see e.g. Figure 4. Furthermore, the proportion of active, passive, and dead peers are very close to the expected values. Recall that by the result of Prop. the ratio of peer types is determined by the eigenvector $\mathbf{s}_1$ corresponding to the maximal eigenvalue γ_1 which is calculated to be about 6:5.3:88.7 [%]. At the end of simulation this ratio is actually registered as 6.1:5.4:88.5 [%].

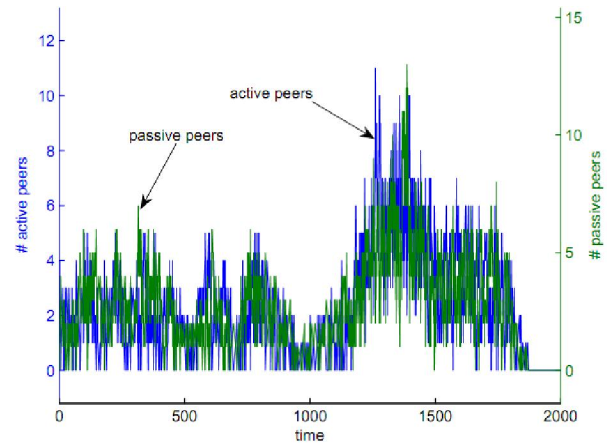


Figure 5. The critical case: γ_1=γ_3=0

It is also worth to showing a critical case when γ_1=0 ($\pi_1 = 0.05$, $\pi_2 = 0.05$; $p_{11} = 0.05$, $p_{12} = 0.9$). This is the case when the process is likely to die out but the extinction time can be very large. As seen in Figure 5 the simulated time is almost 1000. Note that average age time is α=4. The reason is that the expected number of active and passive peers is constant while the expected number of dead peers grows linearly (see Eq. (23)).

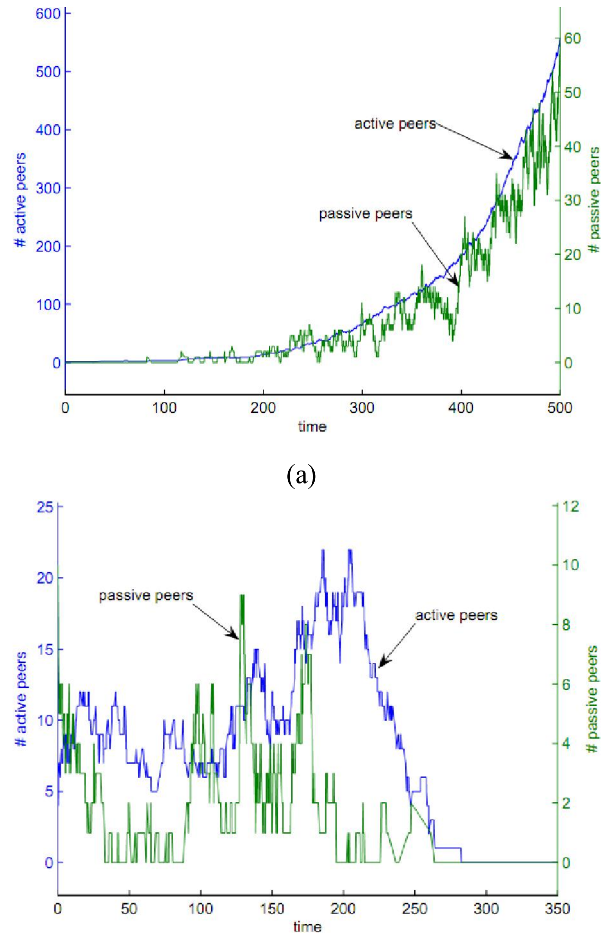
Recent results in the research of P2P systems [11-13] claim the important effects of free riding (peers that do not share). However, our analytical result shows that while free riding is an important factor in P2P system performance, it is not necessarily the only one that determines the system behavior. For instance, set $\pi_1 = 0.05$, $\pi_2 = 0.05$; $p_{11} = 0.95$, $p_{12} = 0.02$, $p_{21} = 0.65$. In this case 90% of the downloads are free riders but with the proper setting of the other parameters the system capacity still grows, see Figure 6(a). In contrast, in another case ($\pi_1 = 0.7$, $\pi_2 = 0.05$; $p_{11} = 0.2$, $p_{12} = 0.1$, $p_{21} = 0.6$), Figure 6(b), when only 25% of the downloads is performed by freeloaders the system still collapses after a finite lifetime.

Finally, in Figure 7 we present a case study when the system parameters are changed during the system operation. Originally the system has the parameter set $\pi_1 = 0.16$, $\pi_2 = 0.04$; $p_{11} = 0.5$, $p_{12} = 0.45$, $p_{21} = 0.75$, which implies that the population will grow to infinity (γ_1=0.0047). This growth can be seen in the left half of the figure. At time $t_0 \simeq 650$ we modify some parameters such that π_1 = 0.04, π_2 = 0.16, while the others are unchanged. The new parameters predict the extinction of the population (γ_1=−0.00039), which is justified by the right side of Figure 7. The system actually died out at about 1350.

B. Practical implications

The proposed branching process model of P2P filesharing systems provides a very clear, simple, and reliable description of the population dynamics of the shared files in the system. The model establishes several practical implications which should be carefully considered by P2P system designers and operators.

- If the population grows, the rate of growth is exponential (Eq. 7).
- Under some certain conditions, the long-term behavior of the system does not depend on several system parameters (see details in Lemma 2 and 5).
- As presented above we argue that the presence of free loaders is not the only factor which determines the system performance. It is one among many other important system descriptors: cooperative peers, online/offline times of peers, age times, etc.
- The model can predict exactly the long-term performance of the system using its set of parameters. A successful system design should apply rules and techniques, e.g. incentives and/or reputation index, which somehow force the possible ranges of system parameters such that the shared files' population grows.
- The results are also valid with different system starting conditions. The impact of new modifications, developments, or any other external circumstances, provisions can be immediately measured, estimated for an ongoing (already under operation) P2P system using a built-in statistical monitor of the software.



(a)

(b)

Figure 6. The effects of free loaders

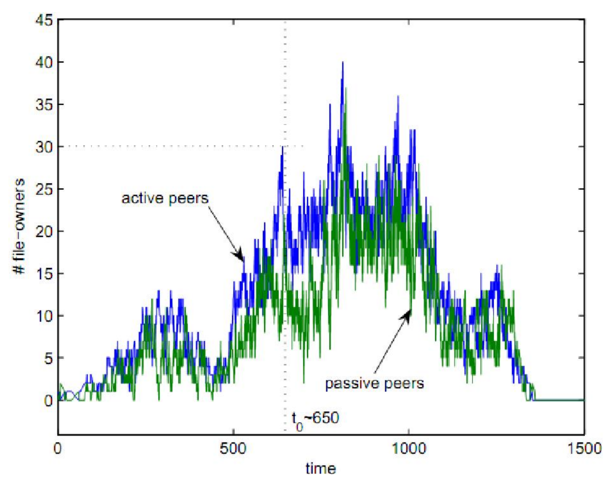


Figure 7. The parameter set is changed at time instant $t_0 \simeq 650$

- In the long term with a fixed combination of parameters the system population dies out or grows exponentially or linearly (see Prop.). The is no other possibility. Nevertheless, in practice the system may exhibit short term stationary behavior several times during its lifetime.

V. CONCLUSION

In this paper we presented a mathematical model to capture the main characteristics of file-sharing peer-to-peer systems. Our model is general and flexible enough to be applied for most of the file-sharing P2P applications in current use. Our results clearly predict the long-term dynamics of the population size.

We have shown that with fixed values of the parameter set the file population will either explode or die out. The important conditions that depend on the system parameters and that determine which case will happen are derived. We also derived the ratio of active, passive and dead peers in the long-term regime and showed that the growth is exponential. Using our results we have found some important practical implications: the population can explode even if most of the peers are free loaders, and the population can become extinct even if most of the peers are cooperative. We can conclude that the free loaders are not the only factor which determines the system behavior.

We proposed some useful guidelines which can help the design and control of such systems since according to our results one can control and predict the system behavior in the future.

However, the presented study still has some limitations. It approaches the P2P systems from the behaviors of a single file in the system, thus cannot explain exactly the performance of the system which contains thousands of individual files. Further, during the life cycle of a file, the users' interest for it is changing and the interest is different from file to file. Therefore the parameter set shall be a function of time and file interest types. These issues are the topics for the future research.

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