SDP Formulation for Design of 2-D Rectangular-Shaped Filters

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***Abstract*: We prose a new technique for the design of linear phase 2-D FIR filter subject to frequency mask constraints, which can handle filters large enough for practical applications. The designed 2-D filters will admit a fast digital implementation. Moreover, we generalize the trigonometric Markov-Lukacs theorem to enable the 2-D filter specifications to be expressed as semi-definite constraints. As a result, the 2-D filter design problem is then posed as semi-definite program (SDP). In addition, we exploited convex duality to derive equivalent SDP formulation of reduced constraints. Design of 2-D rectangular-shaped filters illustrating the advantages of our method is presented.**

Keywords: Non-separable filter, 2-D filter, semi-definite programming, 2-D trigonometric polynomials.

I. Introduction

TWO-DIMENSIONAL (2-D) digital filters and filter banks have found applications in many different fields and are the subject of intensive research (see e.g. [1]–[5] and the references therein). In practice the specifications for a filter are often described by its frequency response, which results in a set of semi-infinite constraints. The design of 2-D filters with frequency response constraints is challenging in both theory and practice. The theoretical challenge concerns the finite dimensional characterization of the 2-D semi-infinite constraints, while the practical challenge concerns the development of numerical tools for realistic applications.

Over the years, numerous 2-D filter design methods have been developed [1], [6], [7], [8], [9], [10], [11]. There are at least three design approaches for 2-D filters:

- Separable filters [12] as products of two 1-D filters are constrained in the limited freedom in design variables and in rectangular spectrum divisions. The pass-band tolerance of the 2-D filters is thus accumulated from that of two 1-D filters, while the ripples in stop-band are taken as the maximum ripples in the corresponding regions of two 1-D filters [13].
- Nonseparable filters are preferred but their design is more difficult and the implementation complexity is much higher. They can be still designed based on transformation from 1-D filters to 2-D filters (see e.g. [1], [3], [7], [9], [10], [12], [14]–[16]). The advantage of 1-D filter based 2-D filters are possibility for fast implementation. A salient approach is the McClellan transform, which maps the frequency points of a 1-D FIR filter into the frequency contours of the designed 2-D filter [14], [16]. However, it is not transparent to control over pass-band and stop-band shapes under the transformation. The accurate cut-off frequency and small error of the designed 2-D filter cannot be guaranteed. In addition, the designed 2-D FIR filters are good only if the cut-off frequency is much less than π and the shape of both the pass-band and stop-band regions are adequately described by the level sets of a unique 2-D filter of low dimension [1]. The design of such low dimensional filters is not quite transparent either. Although an improvement has been made to accommodate the design of wide-band FIR filters in [17], it is still not easy to find a 1-D prototype filter that satisfies the specifications

incorporated from the 2-D pass-band and stop-band requirements.

In comparison with the two above approaches, the frequency sampling design [6], [11] may produce filters of lower dimension with much better pass-band shapes at lower peak ripple but with no available fast implementation. The designed filters are interpolated by the samples on contour lines that match the desired shape of the passband and stop-band. A greater control over pass-band and stop-band can be obtained by placing samples at their edges but it results in increased ripple's size. The total number of contours and numbers of the samples on each contour must be chosen so carefully to avoid singularities [6], [11]. There is only a known necessary (but not sufficient) condition for non-singularity [6] and there is no systematic way for choosing these numbers to avoid singularity. The transition band must be chosen large enough to avoid numerical instability and singularity for interpolation.

An issue that is often overlooked in filter design for both 1-D and 2-D cases is how to efficiently handle the semi-infinite constraints on the ripples at the stop-band and pass-band. The standard approach is to approximate them by finite number of linear over-constraints given at griding points in frequency [18]. There is a certain trade off between the approximation accuracy and numerical stability which can not be easily resolved. The nonuniform discretization of [6], [19], [20] is an effective alternative but it is with substantial effort to guarantee the algorithm convergence and substantially re-tailoring to incorporate additional constraints on the filter coefficients.

In recent years, semi-definite programming (SDP) as an alternative tool to exactly express the semi-infinite constraints of 1-D filters, has renewed attention of many authors [21-24]. One of the major issues in the application of SDP is the high dimensionality of the SDP formulations even for the design of moderate size filters [22], [23]. Although an SDP formulation of the special 2-D positive real semi-infinite constraints has been developed in [4], the size of this SDP formulation is so large that it can only be applied to the design of very small 2-D filters. Another SDP formulation, which is sum-of-squares polynomials for 2-D filter design has been proposed in [25], [26]. Some issues such as correlations between cutoff frequency and coefficients of trigonometric polynomials for both passband and stopband descriptions have not been considered and the dimension of each Gram matrix variable used in [25] is a high-order polynomial in the filter-order.

In this paper, we propose a new approach to the design of 2-D filters with good symmetric frequency response and additional flatness constraints. The purpose of the paper is threefold:

- To derive a finite dimensional representation of the 2-D semi-infinite constraints. The filter design problem with fully achievable specifications and accurate cut-off frequencies is posed as a SDP.
- For computation efficiency, to exploit convex duality to reduce the dimension of the SDP to a manageable size so that filters large enough for practical use can be designed using standard SDP solvers.
- Equally importantly, although our approach is not based on 1-D filters, the designed 2-D filters also have fast implementation as well.

In summary, comparing to other mentioned approaches, our approach is fully advantageous in terms of exactness and flexibility in handling all filter specifications, of computational efficiency for numerical solutions, and fast implementation for the digital implementation. We focus in the paper only on filters with rectangular pass-band and stop-band. The design of filters with pass-band of diamond shape, fan shape and ellipse shape has been considered in [27]–[31]. In comparison with those of rectangular pass-band and stop-bands support, these filters require much fewer functions for band shape description but of much higher order and thus need another development for SDP formulation of reduced order.

The paper is organized as follows. In Section 2 we give an explicit (semi-infinite) optimization formulation for the 2-D filter design problem. Then, by using a result from algebraic geometry [32] we show in Section 3, in a very general setting, that the semi-infinite trigonometric constraints can be addressed by SDP. Then dimension reduction techniques for SDP is developed there using a new approach. The SDP of reduced dimension for 2-D filter design is developed in Section 4 using convex duality. The fast implementation for the designed filters is shown in Section 5. Numerical results to verify the viability of our result are presented in Section 6 and concluding remarks are given in Section 7.

The notations used in the paper are standard, except that $\langle A\rangle$ refers to the trace of a square matrix A, so $\langle AB\rangle=\langle BA\rangle$ for any appropriate matrices A and B. By $X\geq 0$ $(X>0)$ we mean a symmetric positive (strictly positive) definite matrix X. One of the main condition of positive definite matrices that will be frequently used in the paper is $\langle XY\rangle\geq 0$ whenever $X\geq 0$ and $Y\geq 0$.

II. OPTIMIZATION FORMULATION FOR 2-D FILTER DESIGN

The objective of this paper is to design a filter with four-fold symmetric frequency response

$$H(z_1,z_2)=z_1^{-n}z_2^{-n}[b_{00}+\sum_{l=1}^{n}b_{0l}(z_2^{l}+z_2^{-l})+\sum_{i=1}^{n}b_{i0}(z_1^{i}+z_1^{-i})+\sum_{i=1}^{n}\sum_{l=1}^{n}b_{il}(z_1^{i}z_2^{l}+z_1^{i}z_2^{-l}+z_1^{-i}z_2^{l}+z_1^{-i}z_2^{-l})]$$

Its frequency response on the unit disc $\{(z_1,z_2)\}=(e^{j\omega_1},e^{j\omega_2}),\ \Omega:=(\omega_1,\omega_2)\in[0,\pi]^2$ is

$$\begin{aligned}H(\Omega)&=\sum_{i=0}^{n}\sum_{l=0}^{n}x_{il}\cos(i\omega_1)\cos(l\omega_2)=\langle XM(\Omega)\rangle\\&=\varphi_n^T(\omega_1)X\varphi_n(\omega_2),\end{aligned}\tag{1}$$

where

$$\begin{aligned}X\quad&=[x_{il}]_{i,l=0}^{n},x_{00}=b_{00},x_{0l}=2b_{0l},x_{i0}=2b_{i0},\\&\quad x_{il}=4b_{il};i,l=1,...,n,\\\varphi_n(\omega_i)\quad&=(1,\cos\omega_i,\cos 2\omega_i,....\cos n\omega_i)^T,\\M(\Omega)\quad&=\varphi_n(\omega_2)\varphi_n^T(\omega_1).\end{aligned}\tag{2}$$

The design of the filter $H(z_1,z_2)$ involves the selection of the matrix X of filter coefficients such that the frequency response $H(\Omega)$ satisfies a given set of specifications. Typically, these specifications include

- Minimal weighted-square error

$$W_p\int_{\Omega_p}|H(\Omega)-1|^2d\Omega+W_s\int_{\Omega_s}|H(\Omega)|^2d\Omega\tag{3}$$

where

$$\begin{aligned}&\Omega_p=[0,\omega_{1p}]\times[0,\omega_{2p}],\\&\Omega_s=\Omega_{s1}\cup\Omega_{s2},\\&\Omega_{s1}=[\omega_{1s},\pi]\times[0,\pi],\\&\Omega_{s2}=[0,\omega_{1s}]\times[\omega_{2s},\pi]\end{aligned}$$

are pass-band and stop-band of $H(z_1,z_2)$ respectively, while $d\Omega=d\omega_1 d\omega_2$. $W_p>0$ and $W_s>0$ are weighting coefficients.

- Pass-band and stop-band peak-errors stay below the respective tolerances $\delta_p>0,\delta_s>0$, i.e.

$$|H(\Omega)-1|\leq\delta_p\quad\forall\Omega\in\Omega_p,\tag{4}$$

$$|H(\Omega)|\leq\delta_s\quad\forall\Omega\in\Omega_s\tag{5}$$

- Flatness constraints to make $H(\Omega)$ regular of $r-$order $(r\geq 2)$:

$$\left.\frac{\partial^s H(\Omega)}{\partial^i\omega_1\partial^j\omega_2}\right|_{(0,0)}=0,\quad i+j=s=1,2,....,r.\tag{6}$$

Generally, the flatness constraints are desirable when the designed filters are used to generate multi-dimensional wavelets or filter the artifacts between blocks in image processing.

Clearly, the objective (3) is a convex quadratic function in $X\in R^{(n+1)\times(n+1)}$, while (4)-(5) are semi-infinite trigonometric constraints in X. In particular, under a simple arrangement, the above 2-D filter design problem can be rewritten as

$$\begin{aligned}\min_{X\in R^{(n+1)\times(n+1)}}\quad&W_p\langle M_{1p}XM_{2p}X^T\rangle\\&+W_s\langle M_{1s1}XM_{2s1}X^T\rangle\\&+W_s\langle M_{1s2}XM_{2s2}X^T\rangle\\&-2W_p\langle M_pX\rangle\end{aligned}\tag{7}$$

subject to

$$-\delta_p \le \langle XM(\Omega)\rangle - 1 \le \delta_p \quad \forall \Omega \in \Omega_p, \tag{8}$$

$$-\delta_s \le \langle XM(\Omega)\rangle \le \delta_s \quad \forall \Omega \in \Omega_s = \Omega_{s1} \cup \Omega_{s2}, \tag{9}$$

$$\left\langle \frac{\partial^i M(\omega_1,\omega_2)}{\partial^i \omega_1 \partial^j \omega_2}\Bigg|_{(0,0)} X \right\rangle = 0,\ i+j=1,\dots,r \tag{10}$$

where

$$M_{1p} = \int_0^{\omega_{1p}} \varphi_n(\omega_1)\varphi_n^T(\omega_1)d\omega_1,$$

$$M_{2p} = \int_0^{\omega_{2p}} \varphi_n(\omega_2)\varphi_n^T(\omega_2)d\omega_2,$$

$$M_{1s1} = \int_{\omega_{1s}}^{\pi} \varphi_n(\omega_1)\varphi_n^T(\omega_1)d\omega_1,$$

$$M_{2s1} = \int_0^{\pi} \varphi_n(\omega_2)\varphi_n^T(\omega_2)d\omega_2, \tag{11}$$

$$M_{1s2} = \int_0^{\omega_{1s}} \varphi_n(\omega_1)\varphi_n^T(\omega_1)d\omega_1,$$

$$M_{2s2} = \int_{\omega_{2s}}^{\pi} \varphi_n(\omega_2)\varphi_n^T(\omega_2)d\omega_2,$$

$$M_p = \int_0^{\omega_{2p}} \varphi_n(\omega_2)d\omega_2 \times \int_0^{\omega_{1p}} \varphi_n^T(\omega_1)d\omega_1.$$

and partial derivative in (10) is matrix entry-wised.

As:

$$\frac{\partial M(\Omega)}{\partial \omega_1}\Bigg|_{(0,0)} = \frac{\partial M(\Omega)}{\partial \omega_2}\Bigg|_{(0,0)} = \frac{\partial^2 M(\Omega)}{\partial \omega_1 \partial \omega_2}\Bigg|_{(0,0)} = 0$$

the constraint (10) for $r=2$ for instance, actually follows the conditions

$$\left\langle \frac{\partial^2 M(\Omega)}{\partial^2 \omega_1}\Bigg|_{(0,0)}, X \right\rangle = 0, \left\langle \frac{\partial^2 M(\Omega)}{\partial^2 \omega_2}\Bigg|_{(0,0)}, X \right\rangle = 0 \tag{12}$$

The 2-D semi-infinite constraints (8)-(9) pose the foremost difficulty in the design of 2-D filters. Traditionally, there is a number of approaches to handle generic semi-infinite program (SIP), which involve some forms of approximations [18], [33], [34]. As mentioned, the simplest method is to over constrain the peak pass-band and stop-band errors on a set of finite grid points in Ω_p and Ω_s but may cause the numerical instability. A particular SIP $\langle XM(\Omega)\rangle \ge 0 \quad \forall \Omega \in [0,\pi]$ can be equivalently described by semi-definite constraints of very large dimension [4]. Subsequently, this approach cannot even handle design with very moderate value of n such as $n>5$. In the next section, we will show that semi-definite constraints provide an effective finite dimensional representation of the trigonometric constraints (8)-(9).

III. SDP AS A FUNDAMENTAL TOOL FOR HANDLING 2-D SEMI-INFINITE CONSTRAINTS

The above semi-infinite trigonometric constraints (8)-(9) are particular cases of the general semi-infinite constrain

$$g_0(\Omega) > 0 \quad \forall \Omega \in \{\Omega \in [0,\pi]^2 : g_i(\Omega) \ge 0,\ i=1,2,\dots,m\} \tag{13}$$

where $g_i(\Omega),\ i=0,1,\dots,m$ are trigonometric polynomials in Ω, i.e.

$$g_i(\Omega) = \sum_\alpha g_{i\alpha} \cos\Omega^\alpha, \qquad \cos\Omega^\alpha = \cos(\alpha_1\omega_1)\cos(\alpha_2\omega_2) \tag{14}$$

For instance, the first semi-infinite constraint in (8)

$$-\delta_p \le \langle X, M(\Omega)\rangle - 1 \quad \forall \Omega \in \Omega_p \tag{15}$$

corresponds to the following case of (13):

$$g_0(\Omega) = \langle X, M(\Omega)\rangle - 1 + \delta_p$$
$$g_i(\Omega) = \cos\omega_i - \cos\omega_{ip}, \tag{16}$$
$$g_{2+i}(\Omega) = 1 - \cos\omega_i, i=1,2.$$

To develop semi-definite characterizations of the trigonometric semi-infinite constraints, we introduce the 2-D trigonometric bases

$$\phi_i(\Omega) = (1, \cos\omega_1, \cos\omega_2, \cos 2\omega_1, \cos\omega_1\cos\omega_2, \cos 2\omega_2, \cos 3\omega_1, \dots, \cos i\omega_2)^T \tag{17}$$

and the trigonometric moment matrices

$$M_i(\Omega) = \phi_i(\Omega)\phi_i^T(\Omega) \tag{18}$$

which are obviously positive semi-definite.

The fundamental connection between trigonometric semi-infinite constraints and semi-definite constraint is captured by the following theorem.

Theorem 1: The semi-infinite constraint (13) is fulfilled if and only if $g_0(\Omega)$ admits the following representation

$$g_0(\Omega)=\langle X_0 M_{r0}(\Omega)\rangle+\sum_{i=1}^{m} g_i(\Omega)\langle X_i M_{ri}(\Omega)\rangle \qquad (19)$$

with some r_0, r_i and $X_0 \geq 0,\ X_i \geq 0$ of appropriate dimensions.

Proof: See [30, Appendix 1].

By comparing terms at the both sides of (19) with the same trigonometric powers $\cos\Omega^{\alpha}$ one can easily obtain a linear relationship between the coefficients of g_0 and entries of matrices X_0, X_i. This linear relationship and the semi-definite constraints $X_0 \geq 0,\ X_i \geq 0$ constitute a set of semi-definite constraints that is equivalent to the trigonometric semi-infinite constraints (13).

From a practical view point, the following issues are pertinent to the computational implementation of Theorem 1:

- Generally, the numbers r_0, r_i are not known in advance and they are potentially large;
- The dimensions of the matrices M_{r_0}, M_{r_i}, i.e. $(r_0+1)^2\times(r_0+1)^2$ and $(r_i+1)^2\times(r_i+1)^2$ increase very quickly as r_0, r_i increase. Consequently, the dimensions of the matrix variables X_0, X_i, which are the same as those of M_{r_0} and M_{r_i}, increase so quickly that the dimension of resultant SDP grows beyond the capacity of current SDP solvers.

Regarding the first issue, one can fix the numbers r_0, r_i, in which case Theorem 1 provides a sufficient condition for the trigonometric semi-infinite constraint (13). Similar to other multi-dimensional problems, less conservative sufficient conditions are of great interest. We will address these in the subsequent sections.

To provide the reader with some perspectives on our developments in the subsequent sections, we discuss the dimensionality of the potential SDP arising from the 2-D trigonometric semi-infinite constraint (15). For simplicity, consider n odd, i.e. $n=2k+1$. Based on Theorem 1, the simplest sufficient condition for the trigonometric semi-infinite constraint (15) is

$$\begin{aligned}\langle XM(\Omega)\rangle-1+\delta_p &= \langle X_0 M_k(\Omega)\rangle + \\ \sum_{l=1}^{2}[(\cos\omega_l-\cos\omega_{lp})&\langle X_l M_k(\Omega)\rangle \qquad (20)\\ &+(1-\cos\omega_l)\langle X_{l+2}M_k(\Omega)\rangle]\end{aligned}$$

(the minimal $r_i=k$ are chosen beforehand to make the highest power on the right side of (20) match that of the left hand size, which is n). Accordingly, the dimension of X_i is $(k+1)^2\times(k+1)^2$. The total number of scalar variables in (20) is thus

$$5(k+1)^2((k+1)^2+1)/2,$$

which is already in the order of several thousands for a very modest $n=15$.

Another issue is whether the standard constraint $g_i(\Omega)\geq 0, i=1,2,3,4$ with g_i defined in (16) is the best way to express the constraint $\cos\omega_l \in[\cos\omega_{lp},1]$ because $\cos\omega_1$ and $\cos\omega_2$ are uncorrelated in such expression. In the subsequent subsections, we will further explore the geometric structures of 2-D trigonometric polynomial as well as the correlation of the cosine terms over the box $[\cos a,\cos b]\times[\cos c,\cos d]$ to obtain low dimensional semi-definite representations of the semi-infinite constraints (8).

We have raised a number of practical difficulties in using SDP to handle the 2-D trigonometric semi-infinite constraints. Recently, Tuan *et al.* [24] resolved completely the difficulties in 1-D trigonometric semi-infinite constraints through LMI characterizations. In the 1-D case, the numbers r_0, r_i are determined via the trigonometric Markov-Lukacs theorem, and the dimension of the matrix variables X_0, X_i can be efficiently reduced using convex duality. The trigonometric Markov-Lukacs theorem provides an exact representation of a nonnegative trigonometric polynomial by a sum of squares of trigonometric polynomials. By applying singular value decomposition (SVD) to the positive definite matrices X_0, X_i and recalling the definition of trigonometric moment matrices (18), one can see that Theorem 1 provides a means to represent $g_0(\Omega)$ by a sum of squares (SOS) of trigonometric polynomials.

The SOS representation for algebraic polynomials is the subject of intensive research in optimization and control (see e.g. [35], [36]). In this section and section 5, we will address the practical questions of richness and efficiency (in terms of dimension) of SOS representations for both algebraic and trigonometric polynomials, which are still largely open. In particular, it will be shown that the geometric structure of 2-D trigonometric polynomials can be exploited for efficient semi-definite representation of 2-D trigonometric semi-infinite constraints.

The starting point of this section is the following adaptation of Theorem 1.

Theorem 2: **(Generalized 2-D trigonometric Markov-Lukacs theorem)** For $p=(a,b,c,d)\in R^4$ and the function $T:[-1,1]^2\times[0,\pi]^2\to R$,

$$T(\alpha,\beta,\Omega)=(\cos\omega_1-\alpha)(\cos\omega_2-\beta)$$

Let

$$T_{p1}(\Omega)=T(\cos a,\cos c,\Omega), T_{p2}(\Omega)=T(\cos b,\cos d,\Omega),$$
$$T_{p3}(\Omega)=-T(\cos a,\cos d,\Omega), T_{p4}(\Omega)=-T(\cos b,\cos c,\Omega).$$

For a 2-D trigonometric polynomial $g_0(\Omega)=\sum_{i=0}^{n}\sum_{l=0}^{n}g_{il}\cos(i\omega_1)\cos(l\omega_2)$ of order n, one has $g_0(\Omega)\ge 0\ \ \forall\cos\Omega\in[\cos a,\cos b]\times[\cos c,\cos d]$ if it admits the representation

$$g_0(\Omega)=\sum_{i=1}^{4}T_{pi}(\Omega)\langle X_i\Psi(\Omega)\rangle \tag{21}$$

for some $X_i\ge 0$ and $\Psi(\Omega)\ge 0$ $\forall\Omega\in[0,\pi]^2, i=1,2,3,4.$

Proof: See [30, Appendix 2].

The main issue in the application of the above theorem is the construction of the matrices $\Psi(\Omega)$. In the 1-D case, the above theorem provides a necessary and sufficient condition for the non-negativeness of trigonometric polynomial on the interval $[\cos a,\cos b]$ (see [21], [24]), with $\Psi(\omega)\equiv \mathrm{T}_k(\omega)$, $k=[n/2]$ constructed from rank-1 moment matrix

$$\mathrm{T}_k(\omega)=\varphi_k(\omega)\varphi_k^T(\omega), \varphi_k(\omega)=(1,\cos\omega,...,\cos(k\omega))^T$$

i.e. the polynomial $g_0(\omega)=\sum_{i=0}^{n}a_i\cos(i\omega)$ is nonnegative on $[\cos a,\cos b]$ if and only if it admits the following form for some $X_1\ge 0,\ \ X_2\ge 0$:

$$g_0(\omega)=(\cos\omega-\cos a)\langle X_1\mathrm{T}_k(\omega)\rangle + (\cos b-\cos\omega)\langle X_2\mathrm{T}_k(\omega)\rangle. \tag{22}$$

Now, we introduce the Chebyshev polynomials recursions

$$T_0(\Omega)\equiv 1; T_i(\Omega)=2T_{(i-1)}(\Omega)T_1(\Omega)-T_{(i-2)}(\Omega),\quad i=2,3,...., \tag{23}$$

where

$$T_1(\Omega)=A+B\cos\omega_1+C\cos\omega_2+D\cos\omega_1\cos\omega_2 \tag{24}$$

with prescribed numbers A,B,C,D. For instances, in the simulations in Section VI, $A=-1/2,\ B=1/2,\ C=1/2,\ D=1/2$ of the McClellan transform have been utilized.

Lemma 1: For any i, l the following relation holds true

$$T_i(\Omega)T_l(\Omega)=\frac{1}{2}(T_{(i+l)}(\Omega)+T_{|i-l|}(\Omega)) \tag{25}$$

Proof: See [30, Appendix 3].

Next, define the following base

$$\psi(\Omega)=\begin{bmatrix}1\\ T_1(\Omega)\\ \cdots\\ T_k(\Omega)\end{bmatrix}\begin{bmatrix}1\\ T_1(\Omega)\\ \cdots\\ T_k(\Omega)\end{bmatrix}^T \tag{26}$$

which by (25) is

$$\psi(\Omega)=\frac{1}{2}\begin{bmatrix} 1 & T_1(\Omega) & \cdots & T_k(\Omega) \\ T_1(\Omega) & 1 & \cdots & T_{(k-1)}(\Omega) \\ T_2(\Omega) & T_1(\Omega) & \cdots & T_{(k-2)}(\Omega) \\ \vdots & \vdots & \ddots & \vdots \\ T_k(\Omega) & T_{(k-1)}(\Omega) & \cdots & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & T_1(\Omega) & \cdots & T_k(\Omega) \\ T_1(\Omega) & T_2(\Omega) & \cdots & T_{(k+1)}(\Omega) \\ T_2(\Omega) & T_3(\Omega) & \cdots & T_{(k+2)}(\Omega) \\ \vdots & \vdots & \ddots & \vdots \\ T_k(\Omega) & T_{(k+1)}(\Omega) & \cdots & T_{(2k)}(\Omega) \end{bmatrix} \quad (27)$$

The following proposition is a direct consequence of Theorem 2

Corollary 1: For $p=(a,b,c,d)\in R^4$, let C_p be the cone defined by

$$C_p=\{X\in R^{(n+1)\times(n+1)}: \langle XM(\Omega)\rangle \equiv \sum_{i=1}^{4}\langle X_i[\mathrm{T}_{pi1}(\Omega)\Psi(\Omega)]\rangle, X_i\geq 0\} \quad (28)$$

If $X\in C_p$ then

$$\langle XM(\Omega)\rangle \geq 0, \quad \forall \cos\Omega\in[\cos a,\cos b]\times[\cos c,\cos d] \quad (29)$$

The above result states that the cone C_p is a subset of the $n-order$ 2-D trigonometric polynomials that are nonnegative on $[\cos a,\cos b]\times[\cos c,\cos d]$.

In comparison with the dimension of the variables X_i in (20), the dimension of X_i in (28) is $(k+1)\times(k+1)$, i.e. the substantial dimension reduction has been achieved with the new representation (28). In the next section we will show that the linear constraints in the variables X and X_i in the definition (28) can be further reduced using convex duality.

IV. EFFECTIVE SDP FORMULATION FOR THE FILTER DESIGN BY CONVEX DUALITY

We now get back to the 2-D filter design problem that requires minimizing the quadratic objective (7) over the 2-D trigonometric semi-infinite constraints (8)-(9) and linear constraints (10). Based on Corollary 1, we effectively strengthen the semi-infinite constraints (8)-(9) by the semi-definite constraints

$$\begin{gathered} X-(1-\delta_p)E_1\in C_p, -X+(1+\delta_p)E_1\in C_p \\ X+\delta_s E_1\in C_{s_i}, -X+\delta_s E_1\in C_{s_i}, i=1,2 \end{gathered} \quad (30)$$

where $E_1\in R^{(n+1)\times(n+1)}$ with $E_1(0,0)=1$ and $E_1(i,l)=0$ for $i+l>0$ and

$$\begin{gathered} p=(\omega_{1p},0,\omega_{2p},0), s_1=(\pi,\omega_{1s},\pi,0), \\ s_2=(\omega_{1s},0,\pi,\omega_{2s}). \end{gathered} \quad (31)$$

The optimization problem (7), (10), (30) belongs to the following class of problems

$$\begin{aligned} \min_{X\in R^{(n+1)\times(n+1)}} & \sum_{i=1}^{L}\langle M_{1i}XM_{2i}X^T\rangle-\langle MX\rangle: \\ & A_iX+D_i\in C_i, i=1,2,\ldots,m; \\ & \langle \mathrm{A}_{m+j}X\rangle=0, j=1,2,\ldots,q, \end{aligned} \quad (32)$$

where $C_i\subset R^{(n+1)\times(n+1)}$ are cones of the type (28), and $M_{1i}>0, M_{2i}>0$ and M, A_i are given matrices.

Note that our 2-D filter design problem (7)-(10) for $r=2$ is a particular case of problem (32) with

$$\begin{aligned} & L=3, M_{11}=W_p^{1/2}M_{1p}, M_{12}=W_s^{1/2}M_{1s1}, \\ & M_{13}=W_s^{1/2}M_{1s2}, M_{21}=W_p^{1/2}M_{2p}, \\ & M_{22}=W_s^{1/2}M_{2s1}, M_{23}=W_s^{1/2}M_{2s2}, \\ & M=2W_pM_p, m=6, A_1=A_3=A_5=I_{n+1}, \\ & A_2=A_4=A_6=-I_{n+1}, \\ & D_1=(\delta_p-1)E_1, D_2=(1+\delta_p)E_1, \\ & D_3=D_4=D_5=D_6=\delta_sE_1, \\ & C_1=C_2=C_p; C_3=C_{s_1}, C_4=C_{s_2}, \\ & C_5=C_{s_1}, C_6=C_{s_2}, q=2, \\ & A_7=\frac{\partial^2 M(\Omega)}{\partial^2\omega_1}\Big|_{(0,0)}, A_8=\frac{\partial^2 M(\Omega)}{\partial^2\omega_2}\Big|_{(0,0)}. \end{aligned} \quad (33)$$

The following preliminary result is needed to establish the key result for constraint reduction.

Corollary 2: Suppose that M_1 is a linear operator from the space $R^{(n+1)\times(n+1)}$ of real $(n+1)\times(n+1)$ - matrices to itself.

If

$$\langle XM(\Omega)\rangle = \langle ZM_1(M(\Omega))\rangle \quad \forall \cos\Omega \in [\cos a, \cos b]\times[\cos c, \cos d] \tag{34}$$

for some matrices X and $\dot{Z}$ of dimension $(n+1)\times(n+1)$, then it must be true that

$$\langle X, Y\rangle = \langle ZM_1(Y)\rangle \quad \forall Y = [y_{il}]_{i,l=0}^{n} \in R^{(n+1)\times(n+1)}. \tag{35}$$

Proof: See [30, Appendix 4].

Next, define

$\Theta_i(M(\Omega)) = (T_{ki}(\Omega) - a_i)\Psi(\Omega);\ k = p,s;\ i = 1,2,3,4,$

which obviously are linear operators in $M(\Omega)$ and by Corollary 2, we can also write C_p defined by (28) alternatively as

$$C_p = \{\bar{X} \in R^{(n+1)\times(n+1)}: \langle XY\rangle \equiv \Sigma_{i=1}^{4}\langle X_i \Theta_i(Y)\rangle, X_i \geq 0\}, \tag{36}$$

where Y and $\Theta_i(Y)$ are obtained from $M(\Omega)$ and $\Theta_i(M(\Omega))$ through the variable change

$$\cos\Omega^{\alpha} \rightarrow y_{\alpha} = y_{\alpha_1\alpha_2} \forall\alpha, \tag{37}$$

i.e.

$\cos l\omega_1 \rightarrow y_{l0}, \cos l\omega_2 \rightarrow y_{0l}, \cos i\omega_1 \cos l\omega_2 \rightarrow y_{il}.$

Therefore, the dual cone $C_p^* = \{Y: \langle YX\rangle \geq 0 \ \ \forall X \in C_p\}$ of C_p is easily described by the semi-definite constraint

$$C_p^* = \{Y \in R^{(n+1)\times(n+1)}: \Theta_i(Y) \geq 0, \ ii = 1,2,3,4\}. \tag{38}$$

Lemma 2: Given $M_{1i} > 0, M_{2i} > 0$ and M, let X_{opt} denote the optimal solution to the following convex quadratic optimization in the matrix variable X

$$\min_X f(X) = \min_X \sum_{i=1}^{L}\langle M_{1i}XM_{2i}X^T\rangle - \langle X, M\rangle. \tag{39}$$

Then

$$0 = 2\sum_{i=1}^{L} M_{2i}X_{opt}^T M_{1i} - M,$$

$$f(X_{opt}) = -\sum_{i=1}^{L}\langle M_{1i}X_{opt}M_{2i}X_{opt}^T\rangle,$$

$$f(X) - f(X_{opt}) = \sum_{i=1}^{L}\langle M_{1i}(X - X_{opt})M_{2i}(X - X_{opt})^T\rangle \ \ \forall X. \tag{40}$$

Proof: See [30, Appendix 5].

Now, using the Lagrange multiplier approach, (32) is rewritten as follows

$$\min_X \max_{Y^{(i)}\in C_i^*, \lambda_i \in R}[\sum_{i=1}^{L}\langle M_{1i}XM_{2i}X^T\rangle - \langle MX\rangle - \Sigma_{i=1}^{m}\langle Y^{(i)}(A_iX + D_i)\rangle - \sum_{i=1}^{q}\lambda_i\langle A_{i+m}X\rangle] \tag{41}$$

Based on Lemma 2, the dual of (41) is

$$\max_{Y^{(i)}\in C_i^*, \lambda_i\in R} \min_X[\sum_{i=1}^{L}\langle M_{1i}XM_{2i}X^T\rangle - \langle MX\rangle - \sum_{i=1}^{m}\langle Y^{(i)}(A_iX + D_i)\rangle - \sum_{i=1}^{q}\lambda_i\langle A_{i+m}X\rangle] \tag{42}$$

$$= \max_{Y^{(i)}\in C_i^*, \lambda_i\in R}[-\sum_{i=1}^{m}\langle Y^{(i)}D_i\rangle + \min_X(\sum_{i=1}^{L}\langle M_{1i}XM_{2i}X^T\rangle - \left\langle (M + \sum_{i=1}^{m}Y^{(i)}A_i + \sum_{i=1}^{q}\lambda_i A_{i+m})X\right\rangle)] \tag{43}$$

$$= \max_{Y^{(i)}, X_{opt}, \lambda_i}[-\sum_{i=1}^{m}\langle Y^{(i)}D_i\rangle - \sum_{i=1}^{L}\langle M_{1i}X_{opt}M_{2i}X_{opt}^T\rangle]: Y^{(i)} \in C_i^*; \tag{44}$$

$$2\sum_{i=1}^{L}M_{2i}X_{opt}^T M_{1i} - (M + \sum_{i=1}^{m}Y^{(i)}A_i + \sum_{i=1}^{q}\lambda_i A_{i+m}) = 0. \tag{45}$$

The gap between feasible values of (41) and its dual (44)-(45) is also easily derived. For any feasible solutions X and $(Y^{(i)}, X_{opt}, \lambda_i)$ of (41) and (44)-(45), by Lemma 2, the duality gap is derived as follows:

$$\sum_{i=1}^{L} \left\| M_{1i}^{1/2}(X - X_{opt}) M_{2i}^{1/2} \right\|^2 + \sum_{i=1}^{m} \left\langle Y^{(i)}(A_i X + D_i) \right\rangle \geq 0.$$

At the optimal solution X and $(Y^{(i)}, X_{opt}, \lambda_i)$ the gap is zero, i.e.

$$X = X_{opt}, \left\langle Y^{(i)}(A_i X_{opt} + D_i) \right\rangle = 0, \quad i = 1,2,...,L. \tag{46}$$

It is obvious that (44)-(45) is an SDP because the objective is a convex quadratic functional in $Y^{(i)}, X_{opt}$ and the cones C_i^* are described by the SDP (38). An alternative form for the SDP (44)-(45) is

$$\max_{Y^{(i)}, X_{opt}, \lambda_i, Z_i} - \sum_{i=1}^{m} \left\langle Y^{(i)} D_i \right\rangle - \sum_{i=1}^{L} \left\langle M_{1i} Z_i \right\rangle \tag{47}$$

subject to (38), (45),

$$\begin{bmatrix} Z_i & X_{opt} M_{2i} \\ M_{2i} X_{opt}^T & M_{2i} \end{bmatrix} \geq 0, \quad i = 1,2,...,L. \tag{48}$$

Solving the SDP (44)-(45)/(47)-(48) yields the optimal solutions to the primal and dual optimization problems (41) and (42).

From the definition (36) the primal SDP (32) involves $4m$ semi-definite constraints $X_i \geq 0$ and $m(n+1)^2$ linear constraints expressing the linear relationships between the variables X and X_i. On the other hand, by (38) the dual SDP (44)-(45) involves $4m$ semi-definite constraints $\Theta_i(Y) \geq 0$ and just $(n+1)^2$ linear constraints (45) in convenient form for computational implementation. Therefore, the linear constraints have been substantially reduced with the dual SDP (44)-(45).

In summary, given the specifications of the passband $\Omega_p = [0, \omega_{1p}] \times [0, \omega_{2p}]$ with the peak error δ_p, the stopbands $\Omega_{s1} = [\omega_{1s}, \pi] \times [0, \pi]$ and $\Omega_{s2} = [0, \omega_{1s}] \times [\omega_{2s}, \pi]$ with the peak error δ_s and the flatness order $r = 2$, the design procedure for filters meeting these specifications is as follows:

- Input $\omega_{1p}, \omega_{2p}, \omega_{1s}, \omega_{2s}, \delta_p, \delta_s$, weights W_p and W_s and the order n;
- Compute integrals in (11) and partial differentials (12) for the numerical data of the matrices in (33);
- Solve the SDP (47)-(48) to output $Y^{(i)}$ and X_{opt}.
- Check the complementary condition (46). If (46) is satisfied, accept the output X_{opt} as the matrix of the designed filter's coefficients. Otherwise increase the order n and return back to the first step.

Let us give the rationale behind the final step of the above procedure. The dual SDP (44)-(45)/(47)-(48) is generally feasible but the primal SDP (32) may be infeasible (i.e. there is no filter of the given order n of the class considered in this paper that meets the given specifications). However, as far as the complementary condition (46) is met then X_{opt} of the dual SDP (44)-(45)/(47)-(48) is the optimal solution of the primal SDP (32) and of course the later must be feasible.

V. FAST IMPLEMENTATION OF THE DESIGNED FILTER

One of the attractive features of 1-D-based 2-D filter design is the potential for fast implementation. Although the proposed design is not based on 1-D filters, the filters have fast implementation.

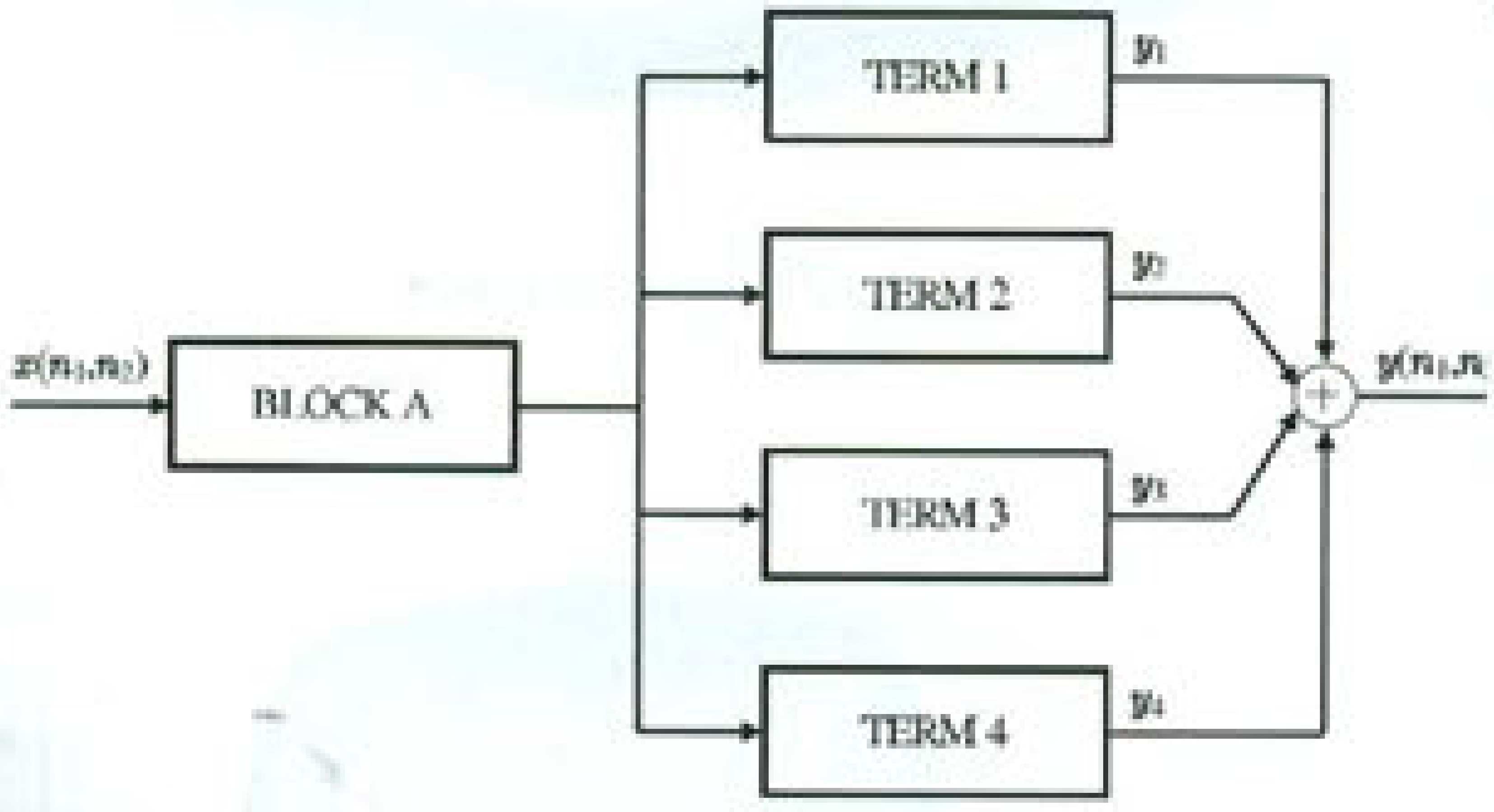
As $\Psi(\Omega)$ defined by (27) is linearly dependent in $T_i(\Omega), i = 1,2,...,n,$ it is straightforward to compute $a_{ji}, j = 1,2,3,4;\ i = 0,1,2,...,n-1$ based on $X_j \geq 0,\ j = 1,2,3,4$ such that

$$H(\Omega) = \sum_{j=1}^{4} T_{pj} \sum_{i=0}^{2k} a_{ji} T_i(\Omega) + (1 - \delta_p). \tag{49}$$

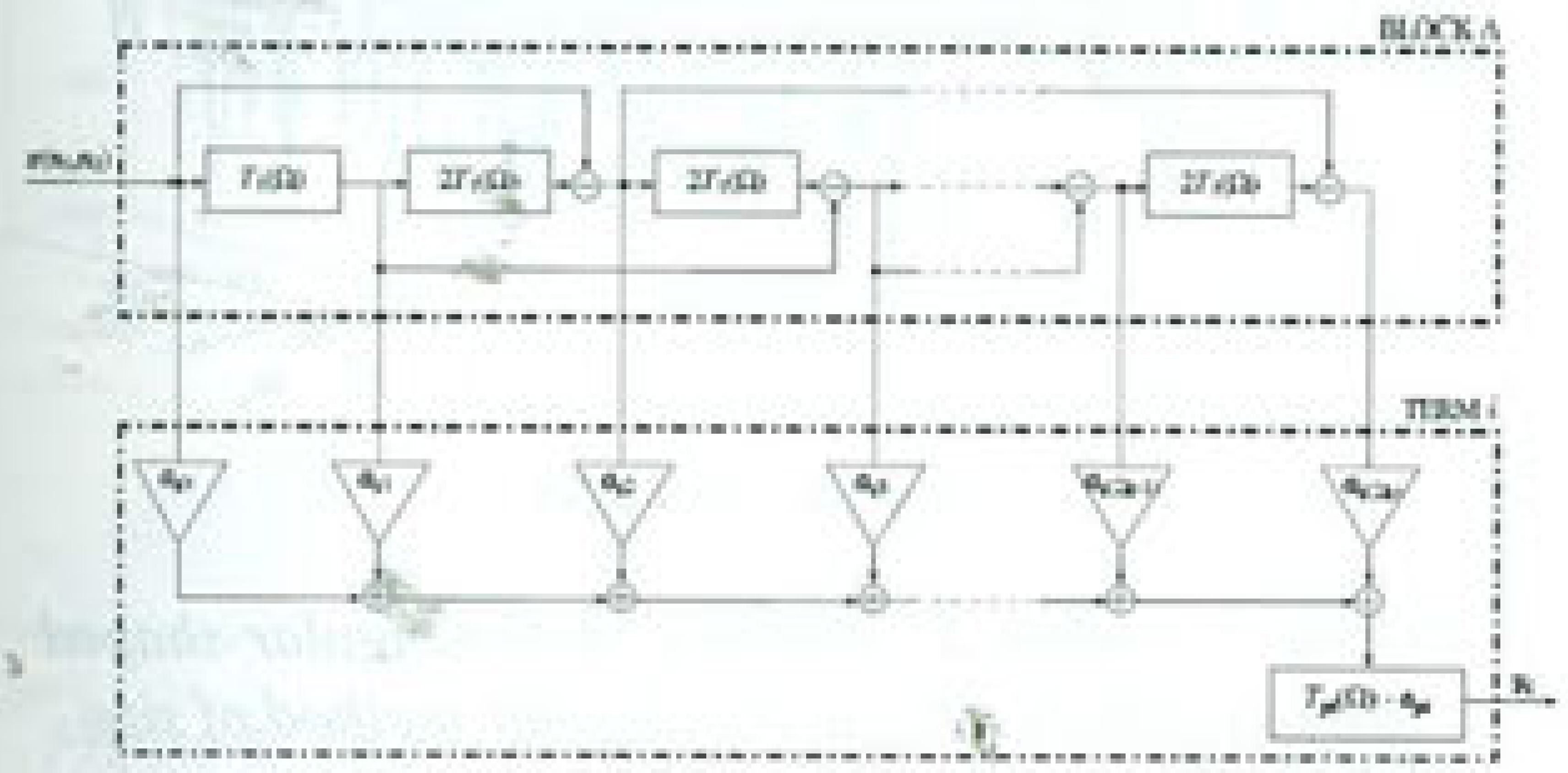
Due to Chebyshev polynomial recursions of $T_i(\Omega)$, each term $\sum_{i=0}^{2k} a_{ji} T_i(\Omega)$ in (49) implies its own fast implementation (Figure 1a). As a result, (49) accumulatively admits fast implementation through the fast implementation for each of its terms. Therefore, the advantages of the Chebyshev recursion are thus fully exploited. The digital implementation for (49) with $k=(n-1)/2$ is depicted in Figure 1. The arithmetic computation for an output point is about $4(n-1)+16$ multiplications and additions. Therefore, the total complexity of the digital implementation for (49) can be easily shown to be

$$2[4n+12] \quad (50)$$

The equation (50) indicates the complexity of our designed filter is proportional to n versus n^2 which is the complexity of a normal 2-D four-fold filter.



(a) Block diagram for the fast implementation of the designed filters



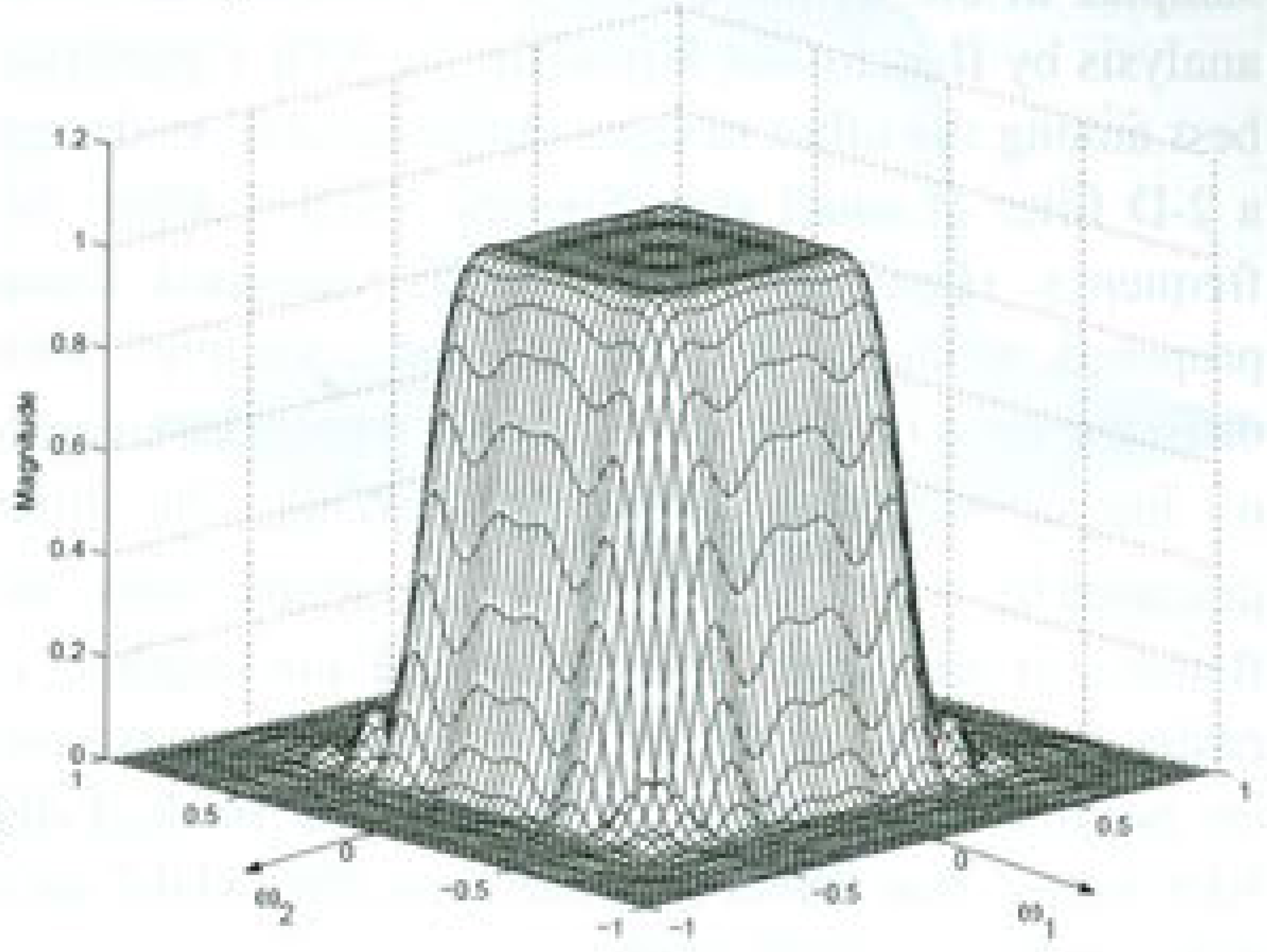
(b) Digital implementation of each term

Figure 1. New structure for 2D digital filter

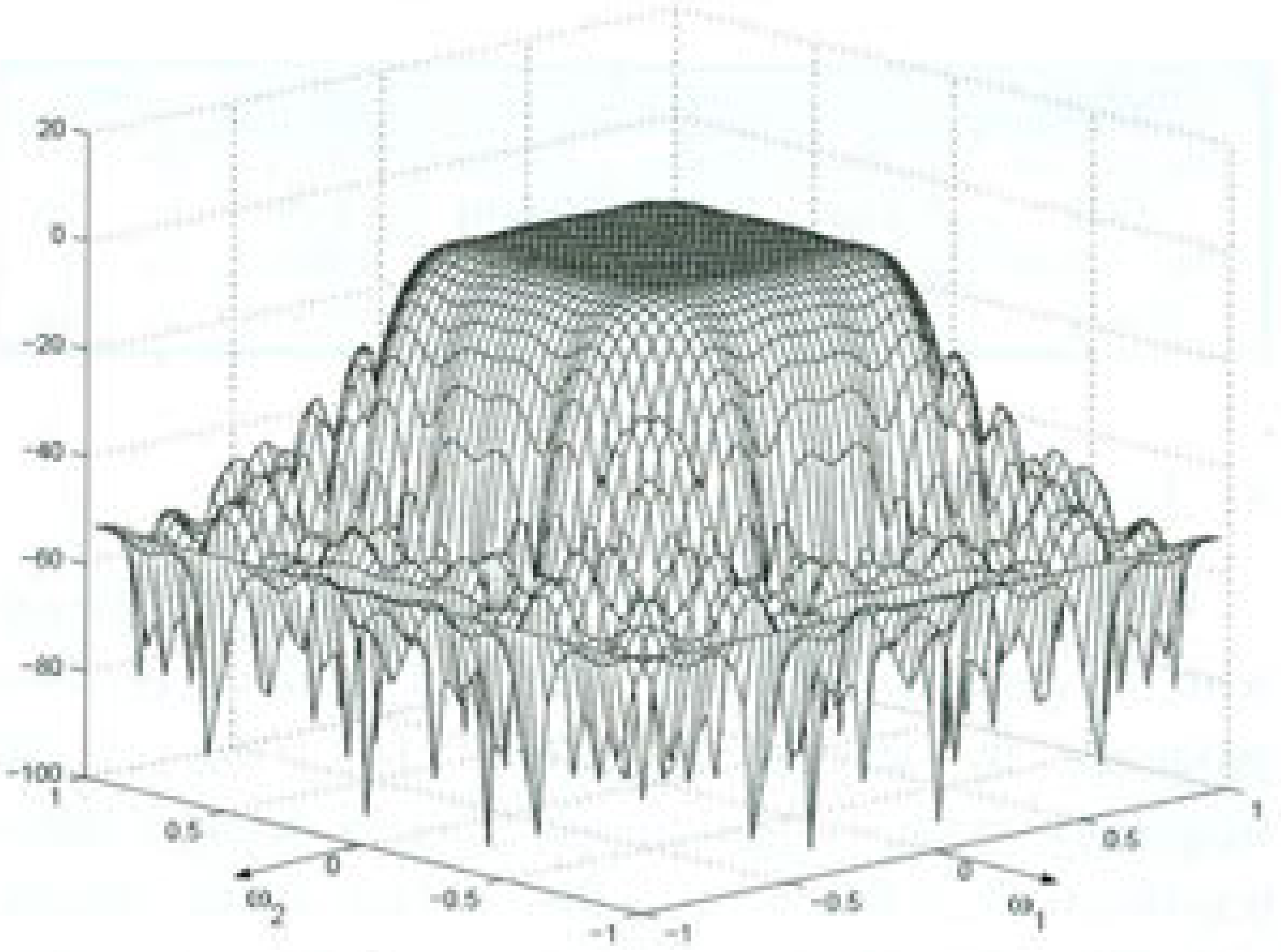
VI. DESIGN OF 2-D REGTANGULAR-SHAPED FILTERS

Two designed examples are shown in this section to demonstrate the effectiveness of our method in comparison with other methods. The simulation was performed using SeDuMi [37], [38] in a MATLAB on a desktop computer (Pentium IV 3.0GHz, 1 GB physical memory). In all examples, the flatness constraint of the second order has been set (i.e. (12) must be satisfied) while as mentioned before, $(A,B,C,D)=(-1/2,\ 1/2,\ 1/2,\ 1/2)$ have been assigned.

A. Example 1



(a) Perspective plot



(b) Decibel plot

Figure 2. Amplitude response of rectangular-shaped low-pass filters designed using NDFT method of size 19 × 19

In this example, we re-design rectangular-shaped lowpass filters of [13, Example 1] using our method and the nonuniform discrete Fourier transform (NDFT) method. Figures 2 and 4 illustrate the frequency responses of these rectangular-shaped lowpass filters designed using the NDFT method. As can be seen, the transition band is worse when the dimensions of the filter increase. It should be noted that the transition band was chosen to be large enough to avoid a possible singularity arising from NDFT design. However, it is very difficult to choose extreme samples in the frequency domain. From the detailed analysis by Bagchi and Mitra [6], the NDFT performs best among the other designs but is suitable to design a 2-D filter of small size. Figures 3 and 5 depict the frequency responses of RS filters designed using proposed method. We choose two examples with different sizes in order to reflect the dimensions parity of matrix X_i in equation (21). While the filter designed by NDFT is clearly not flat at any order, the flatness of the filter designed using our method is easily recognized at the origin. Table 1 demonstrates the performance comparison between our method and NDFT one. For filters of small size, the NDFT is a better choice for filter design.

Table 1. Performance comparison for RS 2-D filter design

Method	ω_p	ω_s	δ_p	δ_s	size
Our method	0.35π	0.65π	0.0192	0.0191	19×19
NDFT	0.35π	0.65π	0.0201	0.0196	19×19
Our method	0.35π	0.65π	0.0095	0.0094	23×23
NDFT	0.35π	0.65π	0.0095	0.0169	23×23

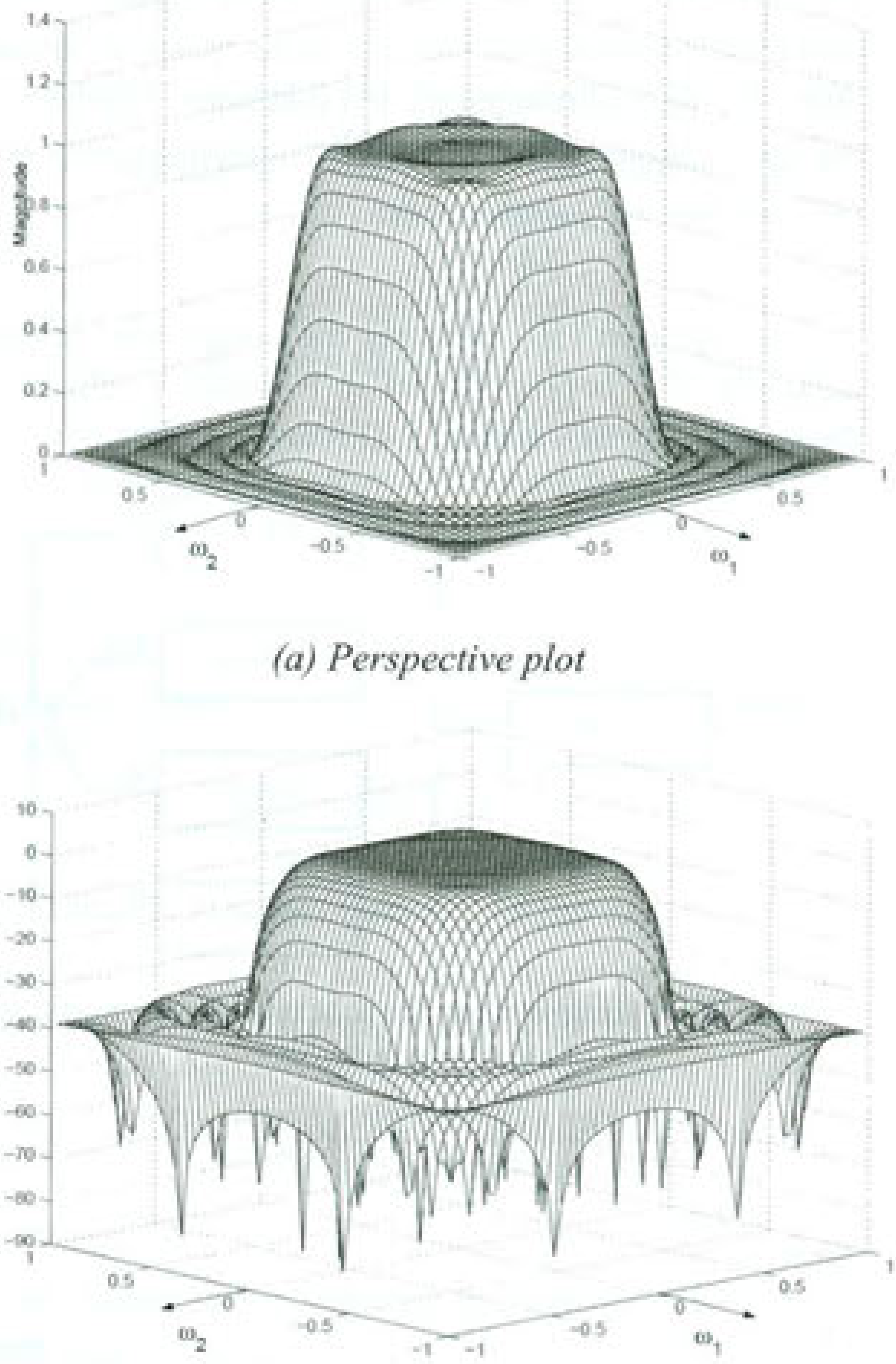
B. *Example 2*

In this second example, we compare our design with a generalization of the McClellan transform proposed by Bagchi & Mitra [16]. The cut-off frequency of the 1-D prototype filter in the McClellan transform must be chosen close to that of the desired 2-D filter. In [16, Example 4.a], a 2-D rectangular-shaped lowpass filter has been designed using the second-order transformation:

$$\cos\omega = \sum_{p=0}^{2}\sum_{q=0}^{2} t(p,q)\cos p\omega_1 \cos q\omega_2, \qquad (51)$$

with:

$t(0,0) = -0.43121,\ t(0,1) = t(1,0) = 0.30365,$
$t(0,2) = t(2,0) = 0.005325,\ t(1,1) = 0.68633,$
$t(1,2) = t(2,1) = 0.008787, t(2,2) = 0.10836.$

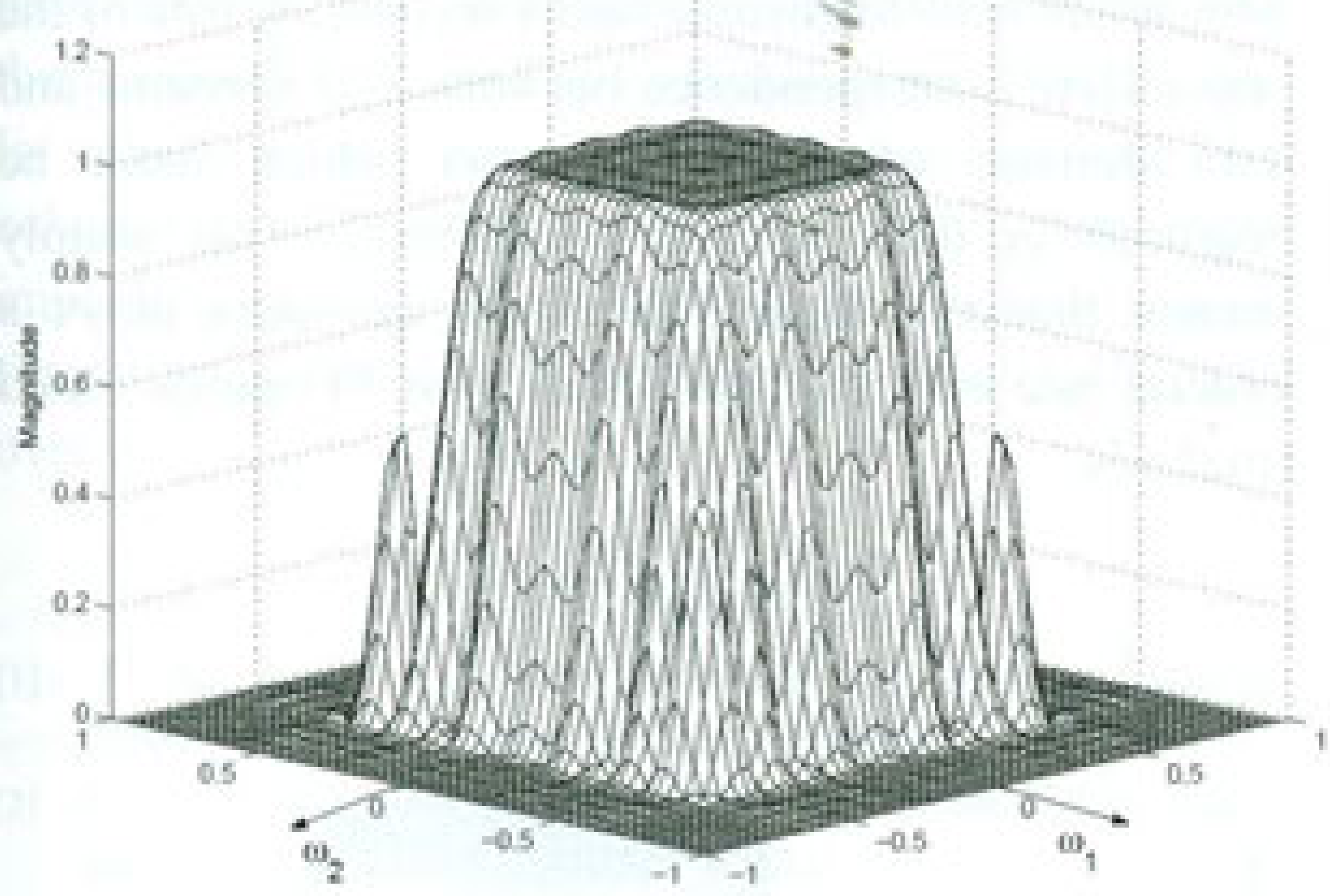


(a) Perspective plot

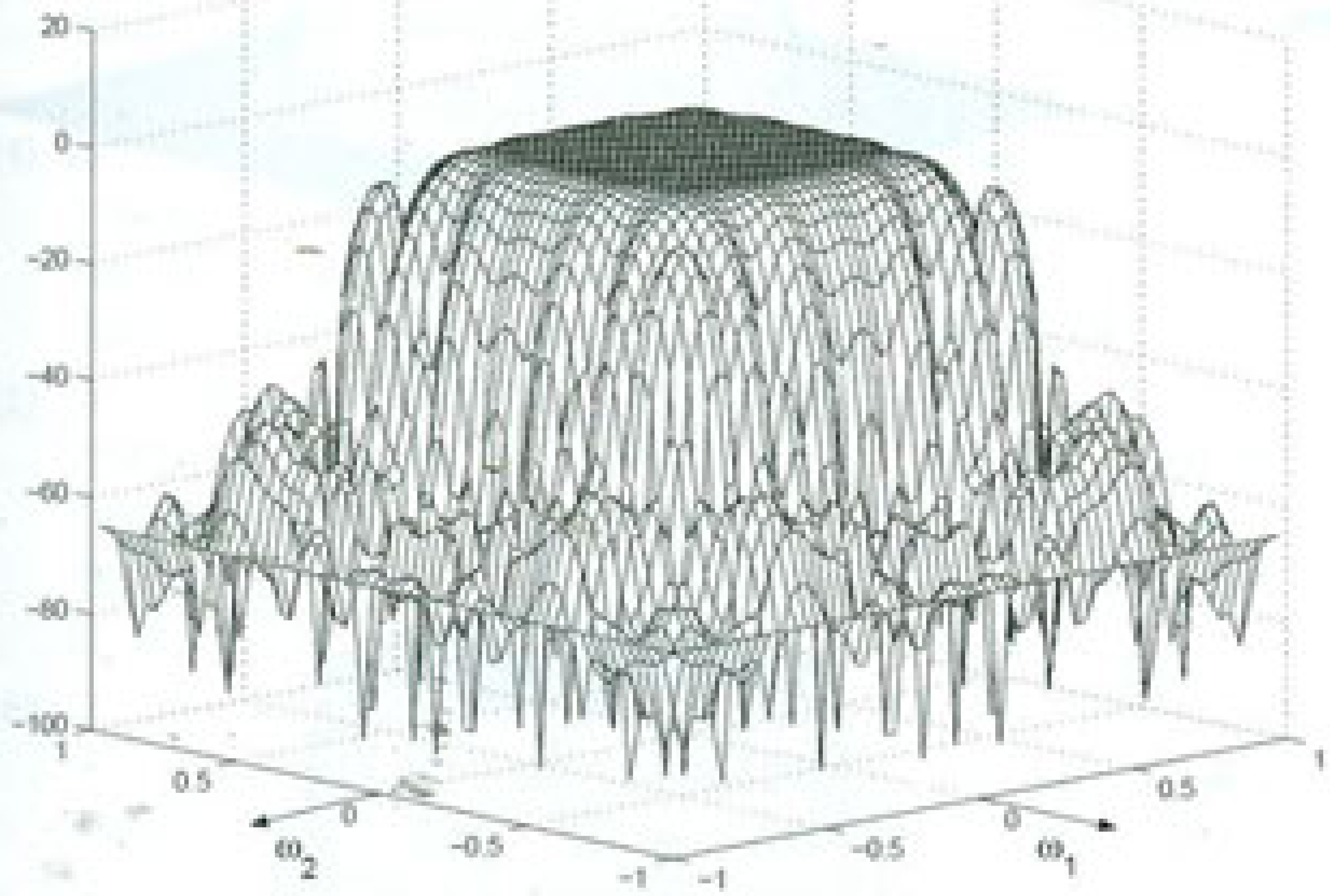
(b) Decibel plot

Figure 3. Amplitude response of rectangular-shaped lowpass filters designed using our method of size 19 × 19.

Due to the second-order transformation, the size of the resultant 2-D filter is doubled in comparison with that of the 1-D filter. Therefore, the rectangular-shaped lowpass filter in [16, Example 4.a] of size 53 × 53 is obtained from a 1-D lowpass filter of size 27. Moreover, to assure an adequate cut-off frequency for the 2-D filter, the 1- D filter must have smaller transition bandwidth. Figure 6(a) shows the frequency response of the resultant 2-D filter with $\omega_{1p} = \omega_{2p} = 0.3\pi, \omega_{1s} = \omega_{2s} = 0.6\pi$, $\delta_p = 0.0513, \delta_s = 0.0199$, through transformation (51).



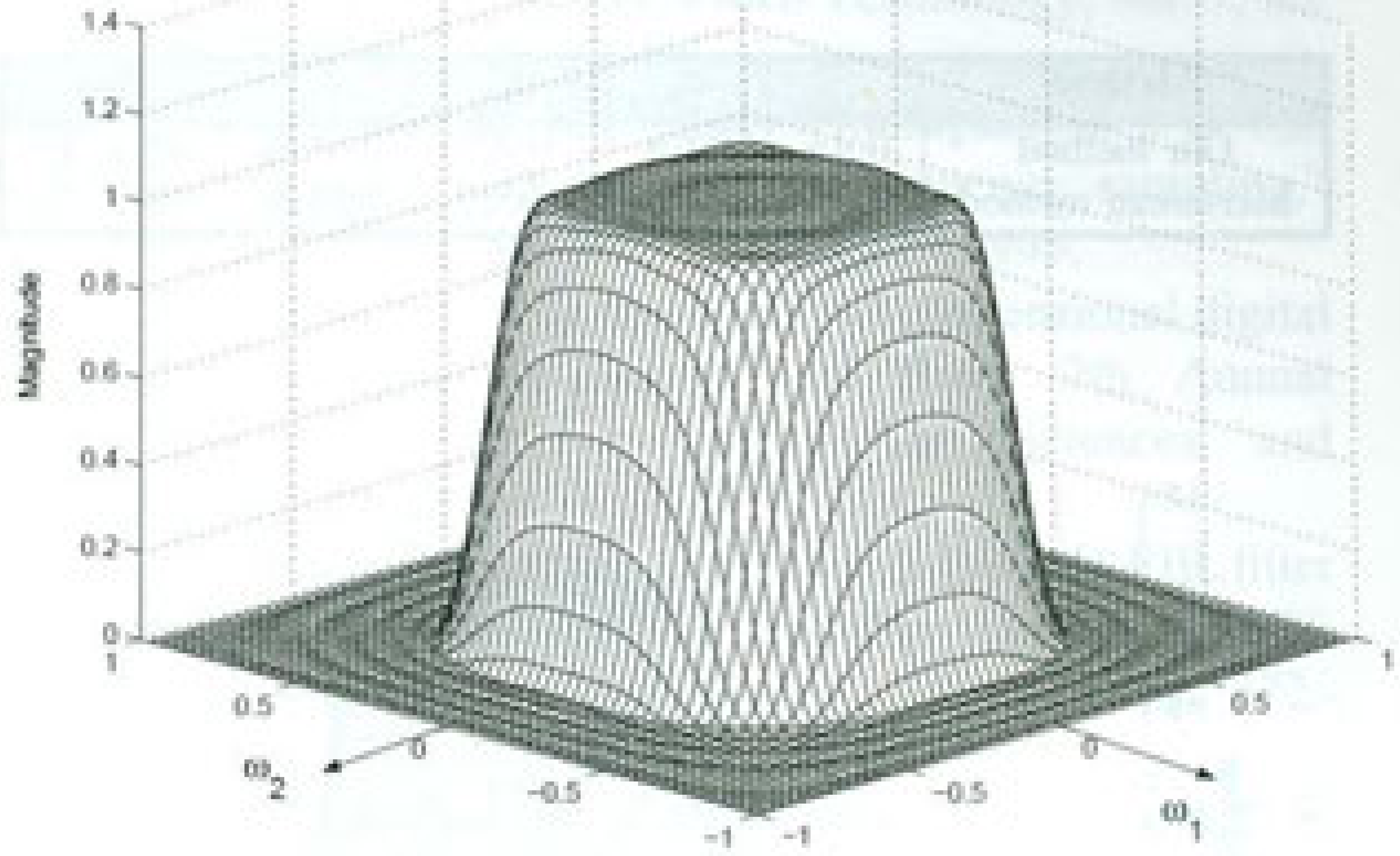
(a) Perspective plot



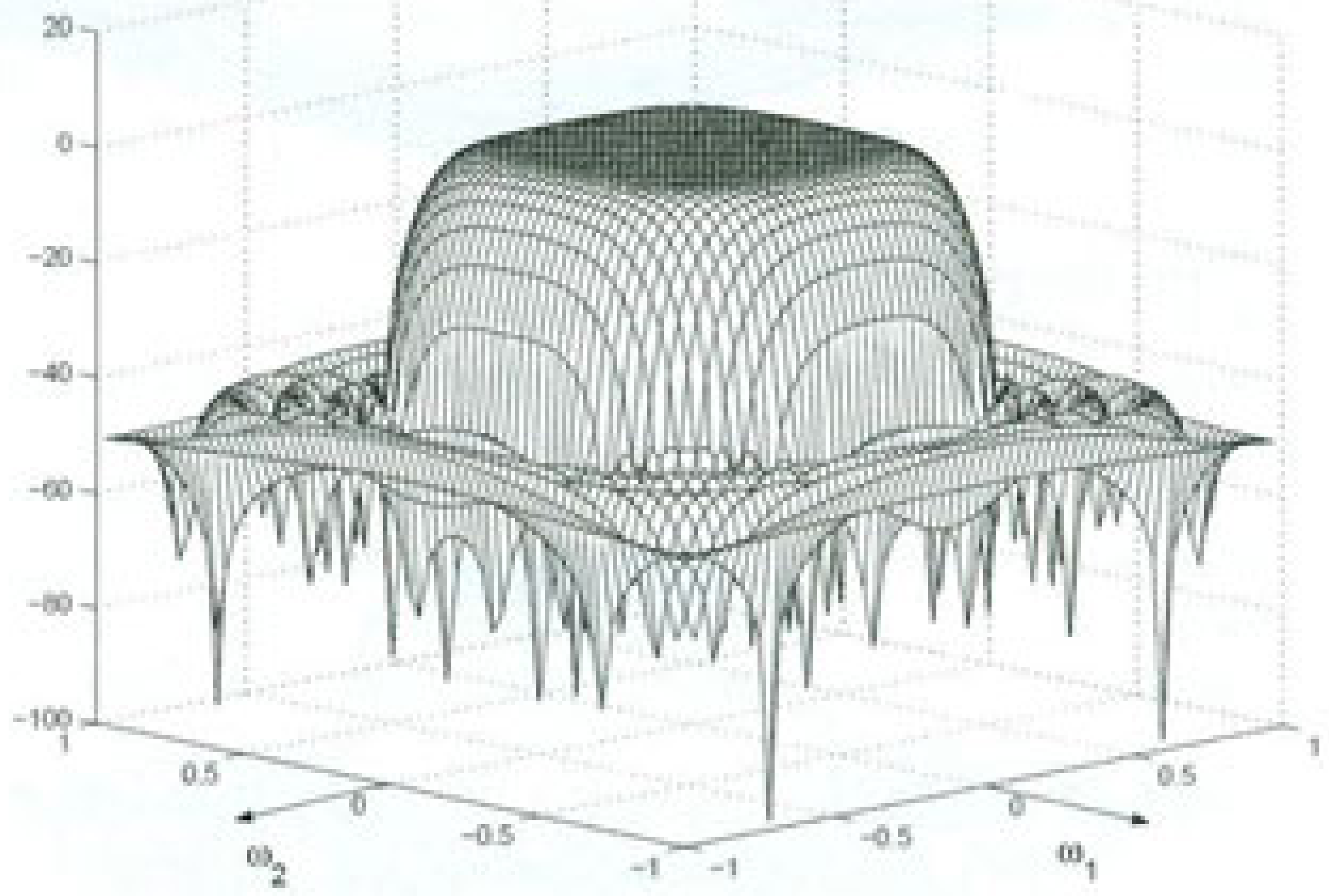
(b) Decibel plot

Figure 4. Amplitude response of rectangular-shaped lowpass filters designed using NDFT method of size 23 × 23.

To obtain this result, the 1-D optimal filter must be designed with $\omega_p = 0.43758\pi$, $\omega_s = 0.56241\pi$, $\delta = 0.001675$. It is clear that the cut-off frequency has been approximated by choosing $\omega_p = 0.43758\pi$ and $\omega_s = 0.56241\pi$, to replace the required conditions $\omega_{1p} = \omega_{2p} = 0.3\pi$ and $\omega_{1s} = \omega_{2s} = 0.6\pi$, respectively. Therefore, the resultant filter is unable to ensure exactly the original specifications.



(a) Perspective plot



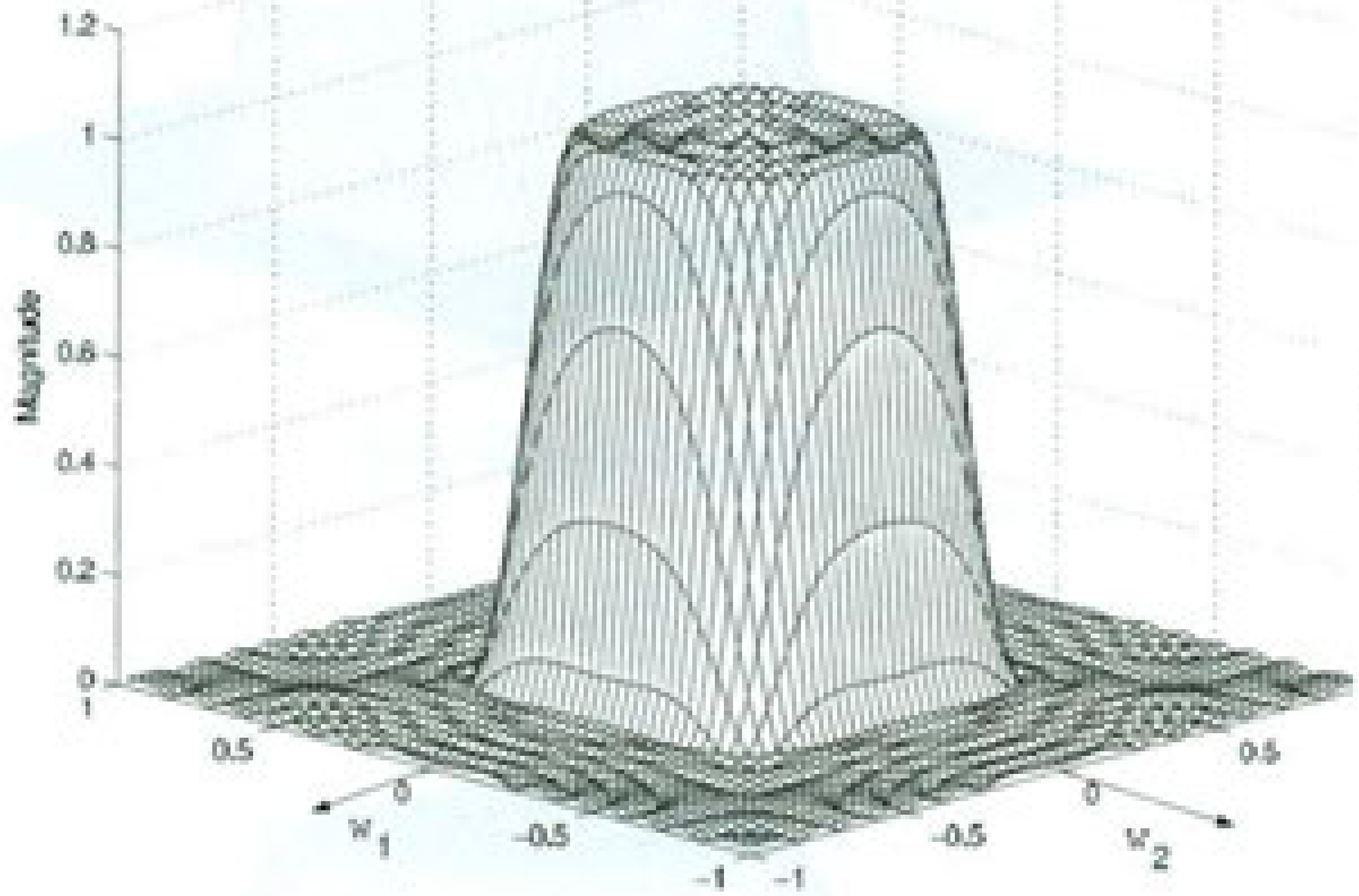
(b) Decibel plot

Figure 5. Amplitude response of rectangular-shaped lowpass filters designed using our method of size 23 × 23.

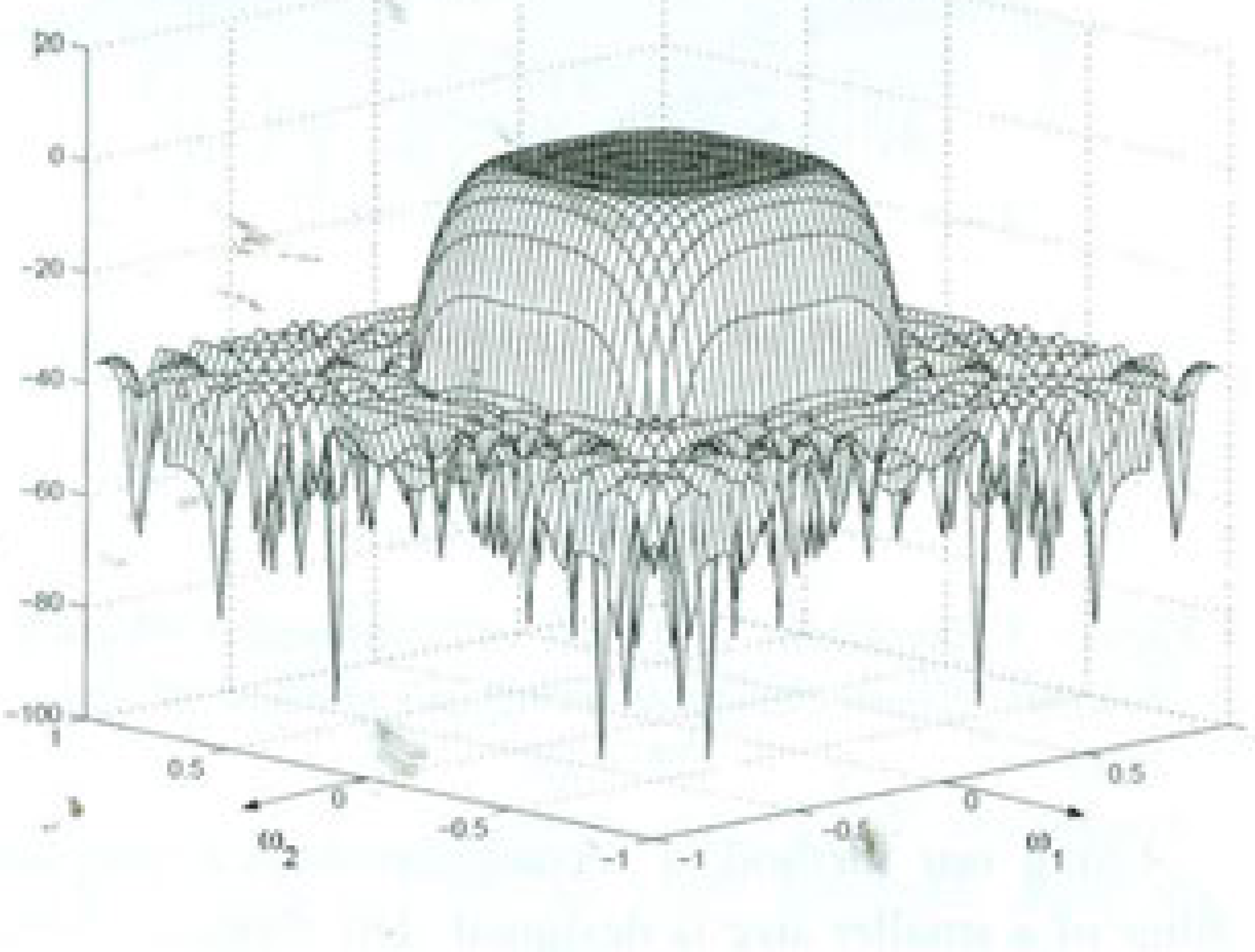
Using our method, a rectangular-shaped lowpass filter of a smaller size is designed. The flatness of the second-order is secured and the accurate cut-off frequency is achieved. Table II illustrates a performance comparison of our filter with the one designed by McClellan transform. Our filter has better attenuation tolerance in passband and stopband even with a smaller transition band and especially with the smaller size. Figure 6 and Figure 7 depict the frequency response of the filter with specifications provided in Table 2.

Table 2. Performance comparison for rs 2-d filter design

Method	ω_p	ω_s	δ_p	δ_s	size
Our method	0.4π	0.6π	0.02	0.0198	43×43
Mersereau method	0.3π	0.6π	0.0513	0.0199	53×53



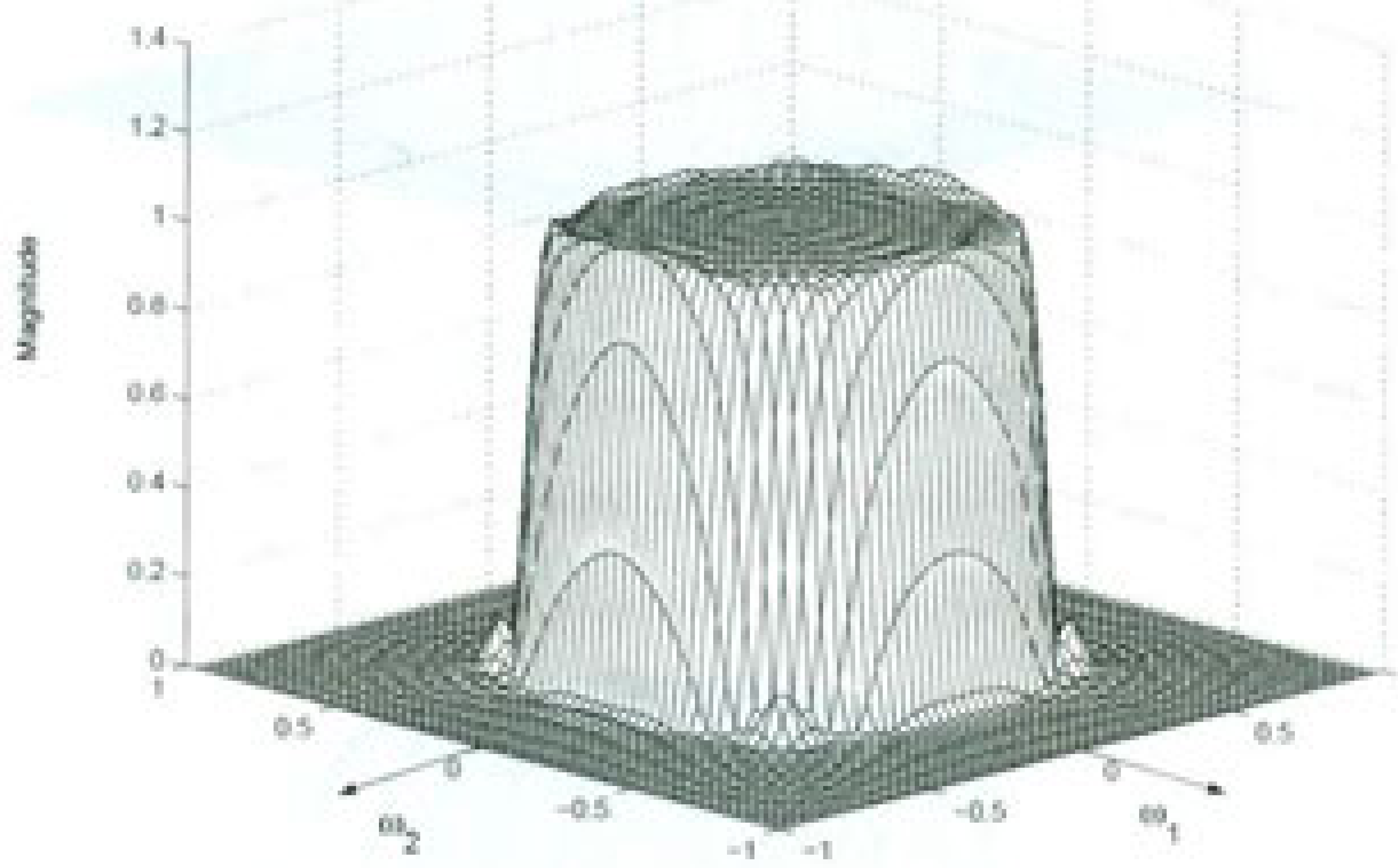
(a) Perspective plot



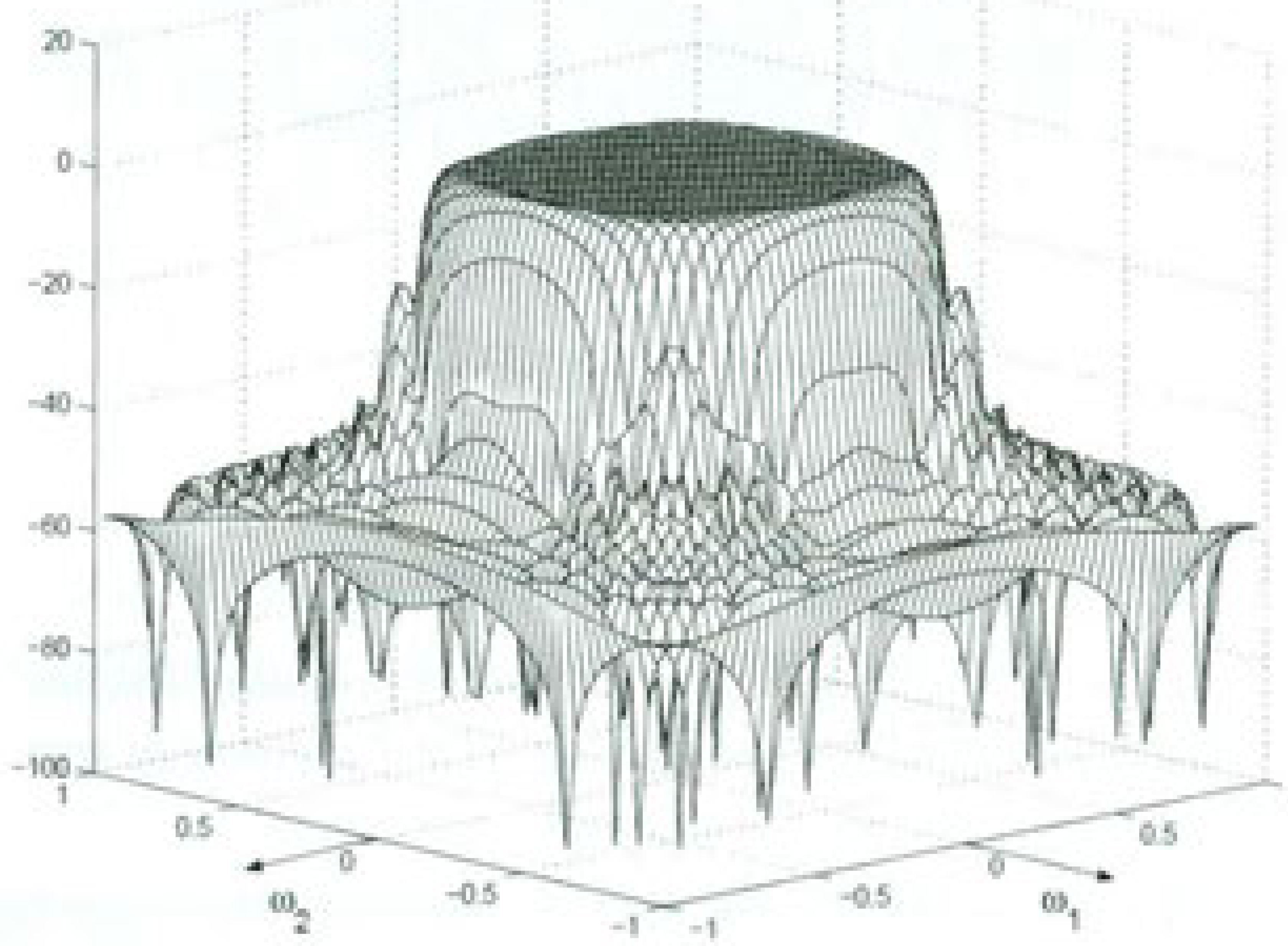
(b) Decibel plot

Figure 6. Amplitude responses of RS lowpass filters designed using the Mersereau method of size 53 × 53.

In the design of a rectangular-shaped filter, the proposed method employs exact descriptions of frequency mask constraints for passband and stopband. Therefore, the shape accuracy of the designed filter is higher, resulting in better performance. The proposed method, in contrast to the McClellan transformation-based method, is free of the one-to-one correspondence between 2-D domains and 1-D domain of cosine function which must be required in the McClellan transform. This simply means that the local concept of contours plays a crucial role only for McClellan trans-formation-based methods.



(a) Perspective plot



(b) Decibel plot

Figure 7. Amplitude responses of RS lowpass filter designed using our method of size 43 × 43.

VII. Conclusion

An efficient technique for the design of 2-D FIR filters with frequency response constraints has been proposed. This technique allows the design of 2-D FIR filters large enough for practical applications to be accomplished by available SDP tools on a standard personal computer. Our SDP formulation ensures that the frequency response specifications are exactly met. In addition, accurate cut-off frequencies can be achieved and the designed filters are fast implemented. The design of 2-D rectangular-shaped filter has demonstrated advantages of our approach over existing methods.

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